Taxable and Tax-Deferred Investing: A Tax-Arbitrage Approach

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We analyze an intertemporal portfolio problem with both taxable and tax-deferred retirement accounts. Using a tax-arbitrage argument, we identify conditions under which the optimal location decision (where to place an asset) is *separable* from the allocation decision (how much to allocate to each asset). Investors place highly taxed assets in the tax-deferred account to maximize the tax benefit and adjust their taxable portfolios to achieve the optimal risk exposure. We show that the two-account problem can be reduced to a taxable-account-only problem. The results are robust to capital gains tax deferrals, consumption and contribution decisions, and stochastic tax rates. (*JEL* G11)

Tax-deferred retirement accounts have grown considerably in popularity in recent years. An investor with significant wealth in both his taxable and tax-deferred accounts faces both the traditional asset allocation decision (how much to allocate to each asset) and a location decision (where to place an asset). When dealing with each account in isolation, the investor's desired allocation is generally different across the two accounts, as he prefers different portfolios under different tax environments.[1] Integrating the two portfolio decisions is clearly important and, not surprisingly, complex, given the potential interactions between these decisions and tax considerations.[2] This paper seeks to shed light

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[1] For example, consider a simple version of the Merton (1969) portfolio problem with continuous time, infinite horizon, and a CRRA investor with risk aversion γ. Let μ be the risky asset's expected return, σ be its volatility, and r be the risk-free rate, and assume that both the risky asset and the bond are taxed upon accrual at the rate τ. Then the optimal percentage of wealth invested in the risky asset is $\frac{\mu-r}{(1-\tau)\gamma\sigma^2}$, which increases with the tax rate τ.

[2] For example, if an investor believes that the stock market is going to outperform bonds, existing theory would suggest that the investor should increase his overall allocation to stocks. With the additional option to invest in the tax-deferred account, one might think that the investor could also benefit from his view by placing more of his stock holdings in the tax-deferred account to reduce his expected capital gains taxes.

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on both the allocation and location decisions for investors facing the choice between taxable and tax-deferred accounts.

In grappling with this task, academics and practitioners have taken rather different approaches. Academics have largely focused on properly capturing the interaction between taxable and tax-deferred decisions. The main approach has been to specify a general framework and to solve the two portfolios simultaneously using numerical procedures.[3] Practitioners have relied on simple rules of thumb that generally ignore the interaction between the two decisions. For example, a practitioner may advocate a two-step approach of first deciding on an overall allocation between asset classes and then placing the most tax-inefficient investments in the tax-deferred account.[4] However, the definition of tax efficiency employed is rather *ad hoc*, and such rules of thumb are largely silent on whether and how investors should modify their overall allocation decision in the presence of tax-deferred accounts.

What is needed is a way to bridge the gap between the two approaches. We achieve this end by showing that, under certain conditions, the taxable and tax-deferred portfolio decisions are separable, and that the overall allocation decision is fully determined by the allocation preference when investors have only taxable accounts. Our results suggest that the two-step approach proposed by practitioners provides a correct framework in which to consider the impact of tax-deferred accounts. However, we also point out that even under the given conditions, the taxable allocation decision needs to be adjusted in order to capture the impact of tax-deferred accounts.

We start with a benchmark setting in which each asset has a different tax rate, and all returns are paid out in full and taxed in each period. We show that the tax-deferred asset returns can be dynamically replicated by a portfolio of taxable assets. The difference between the replication cost and the original tax-deferred cost represents the tax benefit that investors receive for their tax-deferred status. This tax benefit depends only on the tax rates of assets placed in the tax-deferred account, and is independent of expected asset returns or investors' preferences. As a result, investors place only those assets with the highest tax rate in the tax-deferred account to maximize this tax benefit. Therefore, the tax-deferred portfolio fully specifies investors' preferences for asset location, and the location decision is *separable* from the overall portfolio allocation decision. The two-account problem can thus be reduced to a taxable-account-only problem by adjusting the wealth level for the tax benefit: investors can derive their optimal taxable holding by solving for the optimal portfolio associated with the equivalent taxable-account-only problem and subtracting from it the taxable replication portfolios for their tax-deferred holdings.

[3] See, for example, Dammon, Spatt, and Zhang (2004).

[4] See, for example, advice on the Vanguard web site about how to be a "tax-savvy investor."

Next, we extend the model to consider more realistic tax environments, especially the possibility of postponing capital gains taxes. In the context of optimal capital gains realization when investors have access to only a taxable account, Constantinides (1983) shows that the optimal strategy is always to postpone gains and realize losses immediately. We show that this strategy for their taxable holdings continues to be optimal in our setting. Moreover, for investors following this optimal taxable strategy, we can define a unique *current-period effective tax rate* for each asset. All the results derived in the benchmark case remain valid once we use this effective tax rate to rank assets' tax efficiency.

Our model also allows us to characterize the properties of the effective tax rate. Everything else equal, higher coupon bonds, higher dividend stocks, and lower volatility assets have higher effective tax rates. Moreover, individual investors' terminal tax treatment significantly affects the effective tax rates of assets, especially for investors with a shorter investment horizon. While our definition of the effective tax rate is generally consistent with the casual notion of tax efficiency used by practitioners, we provide a systematic way to quantify the tax efficiency of assets. More importantly, we provide theoretical justification for the irrelevance of various factors—such as expected asset returns, subjective views of market conditions, and individual endowments or preferences—for the location decision.[5]

Our paper is most related to the tax-arbitrage literature. Tepper and Affleck (1974); Black (1980); and Tepper (1981) are the first to consider an investor's choice between two accounts that are characterized by different tax treatments. In the context of optimal asset composition within corporate pension accounts, they use a static tax-arbitrage argument to show that if a corporation can issue debt and buy back its own shares in the corporate account, it is always optimal to fully fund the pension account and hold only bonds in it. Titman (1985) extends the tax-arbitrage argument to include forward contracts, and McDonald (2004) considers the tax consequence of a firm issuing options on its own stock.[6] Auerbach and King (1983) point out that this arbitrage extends to individuals making choices between taxable and tax-deferred retirement accounts, and McDonald (2003) extends the arbitrage argument to an employee's decision to exercise stock options.[7]

[5] For example, our results indicate that the proposed intuition in footnote 2 that one's view on stock returns may affect the location choice is in general not correct. The reason is that neither the replication cost of tax-deferred assets nor the effective tax rate is affected by the expected returns of assets.

[6] Titman (1985) finds that a corporation can reap tax benefits by issuing debt and selling forward contracts on its own stock in lieu of issuing equity. McDonald (2004) finds that option positions with implicit borrowing—such as put sales and call purchases—are tax-disadvantaged relative to the equivalent synthetic option with explicit borrowing, while option positions with implicit lending—such as compensation calls—are tax-advantaged.

[7] By exercising the compensation option, the employee can reduce the tax rate on subsequent capital gains at the cost of accelerating the payment of ordinary income tax on current gains. McDonald (2003) finds that it is generally not optimal to do so unless the employee faces portfolio constraints and can borrow to pay the tax.

Another related literature is the dynamic portfolio choice problem with both taxable and tax-deferred accounts. Dammon, Spatt, and Zhang (2004) are the first to address this problem. Using numerical simulations, they solve the lifetime portfolio choice problem with various realistic features, such as labor income, borrowing and short selling constraints, and liquidity considerations, and conclude that the static tax-arbitrage result of preferring higher taxed assets in the tax-deferred account is generally valid in a dynamic setting. Our main contribution is to bridge this literature and the static tax-arbitrage literature by extending the tax-arbitrage argument to the dynamic setting to derive the optimal location result. Moveover, we show that the overall portfolio allocation decision needs to be adjusted to account for the impact of tax-deferred accounts and derive the closed-form expression for this adjustment.

Shoven and Sialm (2003) extend the portfolio choice literature by including stock mutual funds and municipal bonds in the choice set. They find that investors should hold stock mutual funds in tax-deferred accounts and municipal bonds in taxable accounts if mutual funds distribute a high proportion of their total returns as capital gains; otherwise, investors should hold taxable bonds in tax-deferred accounts and mutual funds in taxable accounts. The prediction of our model is consistent with their result if we compare the effective tax rate of all bonds, which is defined as the minimum of the implied tax rate on municipal bonds[8] and the tax rate on taxable bonds, with the effective tax rate of mutual funds, which increases with the proportion of returns distributed as capital gains, to derive the optimal location decision.

The literature on the optimal portfolio allocation problem of an investor with access to only a taxable account is extensive.[9] We complement this literature by addressing a different, but equally important, question: we take as given the optimal strategy identified in this literature and provide a mapping from the optimal strategy for the taxable-account-only problem to that for the two-account problem. For application purposes, investors can use the mapping as an add-on to their existing taxable account strategies. The optimality of our two-account solution is robust to any given single-account strategy. Moreover, the resulting mapping is remarkably simple: we do not need any investor-specific information except for an investor's tax bracket and investment horizon.

Our paper also provides a theoretical benchmark for an expanding empirical literature on whether investors follow tax-efficient strategies when they have access to different accounts with different tax treatments. Using data from the Survey of Consumer Finances, Bergstresser and Poterba (2004) show that about one-third of U.S. investors facing significant location choices could reduce their taxes by reloading heavily taxed assets to their tax-deferred accounts; Amromin (2003) shows that precautionary saving demand for financial

[8] The implied tax rate of a municipal bond is the tax rate that equalizes the return of the municipal bond and the after-tax return of a taxable bond with equal maturity.

[9] See, for example, Constantinides (1983, 1984)); Dybvig and Koo (1996); DeMiguel and Uppal (2005); Dammon, Spatt, and Zhang (2001)); and Gallmeyer, Kaniel, and Tompaidis (2006).

assets, coupled with penalties and restrictions on withdrawing assets from tax-deferred accounts, can partially explain deviations from tax-efficient asset-location patterns; and Amromin, Huang, and Sialm (2007) show that 38% of U.S. households that are accelerating their mortgage payments instead of saving in tax-deferred accounts are making the wrong choice. Using brokerage account data, Barber and Odean (2004) show that investors seem to understand the benefits of locating higher taxed assets in the tax-deferred account, yet do not take full advantage of the tax-arbitrage strategy.

Finally, we contribute to the literature on the impact of retirement accounts on households' saving behavior. It is important to understand the potential differences in the value of a dollar inside and outside tax-deferred accounts in order to properly measure the amount of total savings for any household. Poterba (2004) provides a nice summary of the literature and introduces the concept of *equivalent taxable wealth* for each dollar held in the retirement account. He draws on a range of data sources to calibrate the value of assets held in retirement accounts and describes the differences in relative valuation for households of different ages. Our paper introduces a way to extend the concept of equivalent taxable wealth to a setting in which asset returns are risky.

The paper is structured as follows: Section 1 describes the model setup in the benchmark case and Section 2 presents the optimal portfolio in this case. Section 3 extends the model to consider more realistic tax treatments for investment returns. Section 4 considers other extensions of the model, including the impact of contribution and consumption decisions and stochastic tax rates. Section 5 concludes. Appendix A contains all of the proofs.

1. The Benchmark Model—With Simple Taxation

In this section, we set up a discrete-time, finite-horizon model to solve the optimal portfolio problem when investors hold assets in both taxable and tax-deferred accounts. We make several simplifying assumptions regarding the asset return process, tax environment, and trading constraints to identify the main impact of tax-deferred accounts on portfolio choices.

1.1 Model setup

A risk-free bond (indexed by $i = 1$) and $N - 1$ risky assets (indexed by $i = 2, \ldots, N$) are traded in the economy. Their return distribution is given in the following assumption.

Assumption 1. *Let P_{it} be the ex-dividend price of asset i at time t and $\tilde{r}_{it}$ be the gross return from t to $t+1$, which is deterministic for the risk-free bond, $\tilde{r}_{1t} = r_{ft}$, and risky for all other assets. The net return, $\tilde{r}_{it} - 1$, is paid out in full and taxed at a deterministic rate τ_{it}, which differs both across assets and* *over time. Hence, the after-tax return for asset i is*

$$\widetilde{R}_{it} = \tilde{r}_{it} - \tau_{it}(\tilde{r}_{it} - 1), \qquad i = 1, \ldots, N. \tag{1}$$

This assumption ignores the U.S. tax code's differential treatment of dividends and capital gains, and in particular, the possibility of postponing the realization of capital gains on assets. Instead, it captures different assets' varying degrees of tax efficiency using a single parameter τ_{it}. This reduced-form approach allows us to illustrate the main result in the most straightforward way. We relax this assumption and consider more realistic tax treatments in Section 3. Note further, that by assuming a deterministic tax rate, we are also ignoring potential changes in the tax code over time. We relax this assumption in Section 4.

Our second assumption describes the investment accounts to which investors have access.

Assumption 2. *Each investor has access to both a taxable account and a tax-deferred retirement account. The retirement account is of the Roth type, that is, investors pay taxes on initial contributions but not on future withdrawals.*

In practice, another popular type of retirement account is the 401(k)/403(b) or Keogh plan, in which both the ordinary income taxes on initial contributions and the taxes on interest and capital gains are deferred and taxed as ordinary income only upon withdrawal. Under deterministic tax rates, 401(k) accounts are equivalent to Roth accounts within a constant factor.[10] Given the equivalence, it is sufficient to consider only Roth accounts.

Next we describe investors' utility function and the investment restrictions between the two accounts. Formally, we have the following assumption.

Assumption 3. *Investors have a fixed investment horizon and care only about their utility over terminal consumption, $u(C_T) = u(W_T^T + W_T^D)$, where W_T^T and W_T^D are the terminal wealth in the taxable and the tax-deferred accounts, respectively. Investors can neither contribute to nor withdraw from the tax-deferred account before the terminal date T.*

This utility function assumes that investors can withdraw all wealth from the tax-deferred account at no cost for terminal consumption.[11] The finite-horizon assumption enables us to study explicitly the impact of investment horizons on

[10] The following example illustrates the equivalence between 401(k) and Roth accounts under deterministic tax rates. Let τ_0 and τ_T be the ordinary income tax rates on the initial contribution date and the terminal withdrawal date, respectively, and let $\tilde{r}_P$ be the cumulative before-tax return for any portfolio over T periods. Then the cumulative after-tax return on each dollar of investment in a 401(k) account is $1 \times \tilde{r}_P \times (1 - \tau_T)$, while the return on each dollar in a Roth account is $1 \times (1 - \tau_0) \times \tilde{r}_P$. Clearly, each dollar invested in the 401(k) account is equivalent to $\frac{1-\tau_T}{1-\tau_0}$ dollars in a Roth account, regardless of the realized portfolio return $\tilde{r}_P$.

[11] The value function would be $u(W_T^T + (1 - \tau_T)W_T^D)$ if the tax-deferred account were a 401(k) account. The qualitative results remain the same.

portfolio choices, which is important in the context of retirement savings. Intermediate consumption and the possibility of contributing to or withdrawing from the tax-deferred account before the terminal date are ignored for tractability. We return to these considerations in Section 4.

We now describe the trading environment in each account. For simplicity, we assume that there is no transaction cost in the form of bid-ask spreads, brokerage fees, and so on.

Assumption 4. *There is no transaction cost in either the taxable or the tax-deferred accounts.*

The next assumption describes the trading restriction in the tax-deferred account.

Assumption 5. *Investors are not allowed to borrow or short sell any asset in the tax-deferred account.*

In addition to complying with the current regulation that prohibits short selling in the tax-deferred account, this assumption is also necessary to eliminate tax arbitrage and ensure a properly functioning market. We illustrate this point in Section 2.

Finally, we assume that investors are allowed to short sell in their taxable accounts and we simplify the rules of short selling as follows.

Assumption 6. *Investors are allowed to short sell any asset i in their taxable accounts. They are required to pay the return $\tilde{r}_{it} - 1$ with a full tax deduction at a rate τ_{it}.*

Short selling is rather complicated in the real world, especially given that investors can postpone the realization of capital gains. We defer the detailed discussion to Section 3.

1.2 Individual optimization problem

Under Assumptions 1–6, the only friction is that investors cannot freely move money between their taxable and tax-deferred accounts; within each account, however, they can implement any portfolio strategy at no cost. Hence, the wealth in the taxable account (W_t^T) and the wealth in the tax-deferred account (W_t^D) are the only two state variables at time t. Each investor chooses portfolios in both accounts to maximize utility. Let $\tilde{r}_t = (\tilde{r}_{1t}, \ldots, \tilde{r}_{Nt})'$ and $\widetilde{R}_t = (\tilde{R}_{1t}, \ldots, \tilde{R}_{Nt})'$ be the vectors of before- and after-tax returns, respectively; $\Theta_t^T = (\theta_{1t}^T, \ldots, \theta_{Nt}^T)'$ and $\Theta_t^D = (\theta_{1t}^D, \ldots, \theta_{Nt}^D)'$ be the vectors of portfolio weights in the two accounts after trading at time t; and $J_t(W_t^T, W_t^D)$ be the indirect value function. The dynamic optimization problem for any individual can be written as

$$J_t\left(W_t^T, W_t^D\right) = \max_{\{\Theta_s^T, \Theta_s^D\}_{s=t}^{T-1}} \mathrm{E}_t\left[u\left(W_T^T + W_T^D\right)\right] \tag{2}$$

subject to the following constraints at any time $s = t, \ldots, T-1$:

$$W_{s+1}^T = W_s^T \, \Theta_s^{T\prime} \widetilde{R}_s, \tag{3a}$$

$$W_{s+1}^D = W_s^D \, \Theta_s^{D\prime} \tilde{r}_s, \tag{3b}$$

$$\Theta_s^{T\prime} \mathbf{1}_N = 1, \qquad \Theta_s^{D\prime} \mathbf{1}_N = 1, \quad \text{and} \quad \boldsymbol{\emptyset}_N \leq \Theta_s^D \leq \mathbf{1}_N, \tag{3c}$$

where $\boldsymbol{\emptyset}_N = (0, \ldots, 0)'$ and $\mathbf{1}_N = (1, \ldots, 1)'$ are vectors of 0s and 1s with N elements. Equations (3a) and (3b) give the taxable and tax-deferred accounts' wealth evolutions, respectively. Equation (3c) gives budget constraints for both accounts, together with the borrowing and short-selling constraints for the tax-deferred account (no such constraint is imposed for the taxable account).

1.3 Equivalent taxable wealth

To isolate the impact of tax-deferred accounts, we introduce the concept of *equivalent taxable wealth*,[12] which measures the value of assets in a tax-deferred account relative to assets in a taxable account.

Definition 1. Let $J_t(W_t^T, W_t^D)$ be the value function defined in Equation (2). The *equivalent taxable wealth* is defined as the value $Z_t(W_t^T, W_t^D)$ that solves the following equation:

$$J_t\left(W_t^T, W_t^D\right) = J_t\left(W_t^T + Z_t\left(W_t^T, W_t^D\right)W_t^D, 0\right). \tag{4}$$

Thus, an investor is indifferent between enjoying the privilege of investing in a tax-deferred account and giving up his tax-deferred status while receiving Z_t dollars for each dollar in his tax-deferred account. In general, the value Z_t can only be solved dynamically, since it incorporates information regarding overall portfolio optimization in the future.

2. The Optimal Portfolio in the Benchmark Model

In this section, we solve the optimal portfolio in two steps: first, we use a tax-arbitrage argument to show that the optimal tax-deferred strategy can be solved separately; then we characterize the impact of tax-deferred holdings on the overall portfolio allocation decision.

2.1 The optimal location decision

We start by establishing a link between asset returns in the taxable and tax-deferred accounts.

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[12] Poterba (2004) introduces this concept and quantifies its magnitude in a nonstochastic setting.

Lemma 1 (single-period replication). *Under Assumptions 1–6, the single-period payoff of a dollar of tax-deferred wealth invested in asset i can be replicated by z_{it} dollars of taxable wealth invested in a portfolio of x_{it} dollars of asset i and y_{it} dollars of risk-free bonds, where*

$$x_{it} = \frac{1}{1-\tau_{it}}, \qquad y_{it} = -\frac{\tau_{it}}{(1-\tau_{it})\,R_{ft}}, \quad \textit{and} \quad z_{it} = x_{it} + y_{it}. \tag{5}$$

The lemma can be proved by combining Equation (5) and the after-tax asset returns defined in Equation (1). In particular, the after-tax return of the taxable portfolio (x_{it}, y_{it}) is

$$x_{it}\,\widetilde{R}_{it} + y_{it}\,R_{ft} = \frac{1}{1-\tau_{it}}\big(\tilde{r}_{it} - \tau_{it}(\tilde{r}_{it}-1)\big) - \frac{\tau_{it}}{1-\tau_{it}} = \tilde{r}_{it},$$

which is identical to the return of asset i in the tax-deferred account. Henceforward, we refer to the taxable portfolio (x_{it}, y_{it}) as the *single-period taxable replication portfolio* and z_{it} as the *single-period replication cost* for a dollar of tax-deferred asset i. The fact that $x_{it} > 1$ and $y_{it} < 0$ indicates that holding asset i in the tax-deferred account is equivalent to holding a levered position in asset i in the taxable account financed by risk-free borrowing.

This result is similar to the single-period result of Black (1980). In a dynamic setting, the same $\tilde{r}_{it}$ dollars at $t+1$ face different future investment opportunities in the two accounts. Hence, investors generally have different valuations for a tax-deferred dollar invested in asset i and its single-period taxable replication portfolio. The following lemma identifies the conditions under which tax-deferred wealth can still be dynamically replicated.

Lemma 2 (dynamic replication). *Under Assumptions* 1–6*, if the equivalent taxable wealth in Equation* (4) *is state-independent at time* $t+1$ (*i.e.,* $Z_{t+1}(W_{t+1}^T, W_{t+1}^D) = Z_{t+1}$), *then*

$$J_t\big(W_t^T, W_t^D \mid \Theta_t^D\big) = J_t\big(\widehat{W}_t^T,\ 0\big), \tag{6}$$

where $J_t(\cdot \mid \Theta_t^D)$ is the value function conditional on holding a portfolio Θ_t^D in the tax-deferred account, $\widehat{W}_t^T = W_t^T + (\Theta_t^{D\prime} z_t)\,Z_{t+1}\,W_t^D$, and $z_t = (z_{1t}, \ldots, z_{Nt})'$ for z_{it} defined in Equation (5).

The key assumption is the state-independency of future Z_{t+1}; under this assumption, we can show that each dollar of asset i in the tax-deferred account is equivalent to a taxable portfolio of $x_{it}Z_{t+1}$ dollars of asset i and $y_{it}Z_{t+1}$ dollars of the risk-free bond, with a total replication cost of $z_{it}Z_{t+1}$. Thus, if an investor is committed to holding a portfolio Θ_t^D in the tax-deferred account, each dollar in his tax-deferred account can be replicated by a portfolio costing $(\Theta_t^{D\prime} z_t)\,Z_{t+1}$ dollars in the taxable account. By short selling the replication portfolio in the taxable account (which is feasible under Assumption 6), the investor receives extra $(\Theta_t^{D\prime} z_t)\, Z_{t+1}\, W_t^D$ dollars in the taxable account while perfectly offsetting the risk exposure of his tax-deferred holdings. Hence, he is indifferent between having W_t^D dollars in the tax-deferred account and having $\widehat{W}_t^T$ taxable wealth and no tax-deferred wealth.

It is worth noting that this replication argument does not require that the market be complete with respect to the risk $\tilde{r}_{it}$. In particular, if $\tilde{r}_{it}$ has a continuous distribution between time t and $t+1$, the market is not complete. However, the replication argument still holds, since it only relies on the fact that tax-deferred payoffs are spanned by taxable payoffs.[13]

Given that the value function J_t always increases with wealth in each account, Equation (6) suggests that investors achieve the highest utility when $\widehat{W}_t^T$ is maximized. Intuitively, since investors can always undo any position taken in the tax-deferred account by shorting the replication portfolios in their taxable accounts, the tax-deferred portfolio strategy does not affect the overall risk exposure of investors. Therefore, following different tax-deferred strategies has only a wealth effect on the overall portfolio due to the different equivalent taxable wealth generated, and the optimal tax-deferred strategy is chosen to maximize this equivalent taxable wealth.

From Equation (5), we can see that z_{it} is monotonically increasing in τ_{it}. As long as Z_{t+1} is exogenous, a tax-deferred strategy that holds only assets with the highest tax rate τ_{it} generates the highest $\Theta_t^{D\prime} z_t$ and thus maximizes the value function in Equation (6). To show that the myopic strategy of placing only the highest-taxed asset in the tax-deferred account is in fact optimal in the dynamic setting, we need to verify that the future Z_{t+1} is also maximized and indeed state-independent if we follow this strategy for all future periods. The following proposition summarizes the result.

Proposition 1 (location). *Under Assumptions* 1–6*, the optimal tax-deferred portfolio at any time t is to hold only combinations of assets with the highest tax rate* τ_{it}*, that is,*

$$\Theta_t^D = \sum_{i\in\mathcal{I}_t} \omega_i \boldsymbol{e}_N^i, \quad \textit{where} \sum_{i\in\mathcal{I}_t,\,\omega_i\in[0,1]} \omega_i = 1, \quad \mathcal{I}_t \equiv \{i \mid \tau_{it} = \max_{j\in\{1,\ldots,N\}} \tau_{jt}\}, \tag{7}$$

and $\boldsymbol{e}_N^i = (0,\ldots,1,\ldots,0)'$ *is an index vector of size N that equals* 1 *for the* i*th element and* 0 *otherwise. Moreover, for any investor following the optimal tax-deferred strategy,* $Z_t(W_t^T, W_t^D)$ *in Equation* (4) *is state-independent and can be defined recursively as*

[13] The state-independent tax rate is the main assumption behind this result. As we show in Section 3, when we allow investors to postpone the realization of capital gains, the effective tax rate τ_{it} depends on the realization of asset returns. Then market completeness regarding $\tilde{r}_{it}$ risk is required for the replication result.

$$Z_T = 1, \quad \textit{and } Z_t \equiv \hat{z}_t\, Z_{t+1}, \quad t < T, \tag{8}$$

where $\hat{z}_t = \max\{z_{it} \mid i \in \mathcal{I}_t\}$, *with* z_{it} *defined in Equation* (5).

We prove the proposition using backward induction in Appendix A. Since z_{it} monotonically increases in the tax rate τ_{it}, all assets in the set $\mathcal{I}_t$ have the same z_{it}. The quantity $\hat{z}_t$ is uniquely defined and is equal to the maximum single-period replication cost at time t. Proposition 1 suggests an extremely simple strategy in the tax-deferred account: each period, investors rank all assets by their single-period tax rates and place only those assets with the highest tax rate in the tax-deferred account. If there are several assets sharing the highest tax rate, the optimal tax-deferred portfolio is any linear combination of those assets. The optimal tax-deferred strategy is myopic since no future information is taken into account, other than the fact that investors will be optimizing in the future. Our result thus provides theoretical justification for the common practice of myopically ranking assets by their tax rates and placing higher taxed assets in the tax-deferred account.

Note that Assumption 5 of no borrowing or short selling in the tax-deferred account is imposed to comply with the current tax code. Using the replication argument, we can show that it is also necessary to rule out arbitrage opportunities. To illustrate this, consider two assets with different tax rates, say $\tau_{it} > \tau_{jt}$. Then $z_{it} > z_{jt}$; that is, investing a tax-deferred dollar in asset i yields more equivalent taxable wealth than investing in asset j. If investors are allowed to borrow and short sell in the tax-deferred account, they can short sell one dollar of asset j and long one dollar of asset i in the tax-deferred account, while at the same time short sell the replication portfolio for asset i (yielding $z_{it}Z_{t+1}$ dollars) and long the replication portfolio for asset j (costing $z_{jt}Z_{t+1}$ dollars) in the taxable account. By the definition of replication portfolios, the overall risk exposure is zero. The difference in the replication cost, $(z_{it} - z_{jt})Z_{t+1} > 0$, represents an arbitrage profit. Thus, it is important to impose Assumption 5 in order to ensure a properly functioning market.

2.2 The optimal allocation decision

We now study the impact of tax-deferred holdings on the overall portfolio allocation. Given the vast literature on portfolio choices with only a taxable account, we focus our analysis on how investors should adjust their single-account optimal portfolios in order to account for their asset holdings in the tax-deferred account. In the following proposition, we derive the optimal taxable portfolio, taking as given both the optimal single-account portfolio and the optimal tax-deferred strategy from the previous section.

Proposition 2 (allocation). *Under Assumptions* 1–6, *let* $\Theta_t^*(W_t^T) = (\theta_{1t}^*, \ldots, \theta_{Nt}^*)'$ *be the optimal portfolio when investors have access to only a taxable account,* $\Theta_t^D = (\theta_{1t}^D, \ldots, \theta_{Nt}^D)'$ *be the optimal tax-deferred portfolio* *in Equation* (7)*,* Z_t *be the equivalent taxable wealth in Equation* (8)*, and* $\widehat{W}_t^T = W_t^T + Z_t W_t^D$*. Then the optimal taxable portfolio is* $\Theta_t^T(W_t^T, W_t^D) \equiv (\theta_{1t}^T, \ldots, \theta_{Nt}^T)$*, where*

$$\theta_{1t}^T = \frac{1}{W_t^T}\left(\widehat{W}_t^T\,\theta_{1t}^*\big(\widehat{W}_t^T\big) - \big(x_{1t}\theta_{1t}^D + y_t'\Theta_t^D\big)Z_{t+1}W_t^D\right), \tag{9a}$$

$$\theta_{it}^T = \frac{1}{W_t^T}\left(\widehat{W}_t^T\,\theta_{it}^*\big(\widehat{W}_t^T\big) - x_{it}\theta_{it}^D Z_{t+1}W_t^D\right), \quad i = 2, \ldots, N, \tag{9b}$$

$y_t = (y_{1t}, \ldots, y_{Nt})'$*, and both* x_{it} *and* y_{it} *are defined in Equation* (5)*.*

Proposition 2 follows directly from Lemmata 1 and 2 and Proposition 1. Since having one dollar in the tax-deferred account following the optimal strategy Θ_t^D is equivalent to having Z_t dollars of taxable wealth, the original problem with both taxable and tax-deferred accounts can be reduced to a problem with only a taxable account if we increase the wealth level by $Z_t W_t^D$. In order to derive the optimal overall risk exposure, we solve the single-account problem with a total wealth of $\widehat{W}_t^T = W_t^T + Z_t W_t^D$, yielding the first terms in Equations (9a) and (9b). The second terms correspond to the fact that investors need to subtract the effective taxable risk exposure resulting from their tax-deferred holdings in order to derive their optimal taxable portfolio. From Equation (5) we know that $x_{it} > 0$. Hence, if an investor holds asset i in his tax-deferred account, he reduces his holding θ_{it}^T of the same asset in the taxable account. For sufficiently large $x_{it}\,\theta_{it}^D Z_{t+1} W_t^D$, it is even possible to have $\theta_{it}^T < 0$, implying that an investor may need to short sell asset i in the taxable account to offset his excessive holding of it in the tax-deferred account.

Since investors generally hold assets with higher single-period replication costs in the tax-deferred account and they do not hold (or even need to short sell) these assets in the taxable account, we can view these assets as having a preferred location in the tax-deferred account. All other assets are held solely in the taxable account, and hence can be viewed as having a preferred location in the taxable account. In this sense, the tax-deferred portfolio decision fully specifies whether to hold an asset in the taxable versus tax-deferred account. Accordingly, we say that Proposition 1 corresponds to the *location decision*, whereas Proposition 2, which determines the overall risk exposure, corresponds to the *allocation decision*. Moreover, given that the equivalent taxable wealth is independent of individual preference or overall portfolio decisions, Propositions 1 and 2 also imply that the location decision is fully separable from the overall portfolio allocation decision.

The significance of Propositions 1 and 2 can be illustrated using the following example. Assume there are two investors who differ drastically in their views on assets i and j; in particular, investor 1 (or 2) would like to hold 100% of his portfolio in asset i (or j). Assuming that asset i has the highest single-period tax rate τ_{it} among all assets, then both investors choose to hold the *same* tax-deferred portfolio, namely, 100% in asset i in their tax-deferred accounts. The individual preference is reflected only in the corresponding taxable portfolio, in which investor 2 holds a huge short position in asset i to offset the undesired exposure in his tax-deferred account, while investor 1 holds only asset i to reflect his strong preference for it.

In summary, we are able to derive three main results. First, the optimal location decision is separable from the overall portfolio allocation decision, and can be determined myopically by tax arbitrage. In particular, investors should place only those assets with the highest single-period tax rate in the tax-deferred account. Second, the equivalent taxable wealth for each dollar in the tax-deferred account is determined solely by the risk-free rate and the tax rates of assets placed in the tax-deferred account, and it is independent of investors' risk preferences and other state variables. Finally, the optimal portfolio allocation for the two-account problem can be derived through a preference-free mapping from the two-account problem to a taxable-account-only problem.

While the benchmark model yields interesting results regarding location and allocation decisions, one might be concerned about the robustness of these results given the streamlined nature of the underlying assumptions. In the remaining sections of the paper, we relax Assumptions 1–6 to verify the robustness of our results.

3. Realistic Taxation of Investment Incomes

In the benchmark case, we assume that there is a different tax rate τ_{it} for each asset and that different types of asset returns are taxed equally. In reality, however, the U.S. tax code only specifies different tax rates for different types of returns, such as interest payments, dividend income, and capital gains. In this section, we show that it is possible to construct an effective tax rate for each asset in the more realistic tax environment, and that all the results in the benchmark case continue to hold. In particular, we consider in Section 3.1 the simpler case in which investors realize gains and losses immediately and show that the effective tax rate can be constructed as a weighted average of tax rates on different types of returns. In Section 3.2, we allow investors to postpone the realization of capital gains and show how to construct the effective tax rate for each asset by extending the approach of Constantinides (1983). In Section 3.3, we discuss the properties of the corresponding effective tax rates.

3.1 No tax-timing option on capital gains

We start with the case in which investors are required to realize capital gains immediately. In place of Assumption 1, we assume the following.

Assumption 7. *Let c_{it} be the risk-free coupon rate and $\tilde{g}_{it} \equiv \frac{\tilde{P}_{it+1} - P_{it}}{P_{it}}$ be the capital gains rate. The before-tax return on any asset i from t to t + 1 is*

$$\tilde{r}_{it} = 1 + c_{it} + \tilde{g}_{it}, \qquad i = 1, \ldots, N. \tag{10}$$

Both c_{it} and $\tilde{g}_{it}$ are paid out in full and taxed annually at τ_{ct} and τ_{gt}, respectively. Hence, the after-tax return for asset i can be expressed as

$$\widetilde{R}_{it} = 1 + (1 - \tau_{ct})\, c_{it} + (1 - \tau_{gt})\, \tilde{g}_{it}. \tag{11}$$

The return on most assets can be interpreted as c_{it} and $\tilde{g}_{it}$. For example, the risk-free asset has $c_{it} = r_{ft} - 1$ and $\tilde{g}_{it} = 0$. A long-term coupon bond usually has a fixed coupon payment C_{it}, but the underlying price P_{it} of the bond may change stochastically. We can calculate its coupon rate $c_{it} = C_{it}/P_{it}$ and the capital gains rate for each period. Note that this c_{it} changes over time and is usually different from the specified coupon rate for the bond.[14] For stocks, we can treat dividends as the coupon payments. Since dividends are rather smooth over time, we can assume a deterministic dividend rate. Although the tax rate τ_{ct} on qualified dividends is generally different from that on interest payments, the functional form of Equation (11) is robust. Historically, as shown in Sialm (2006), the tax rate on coupon payments is usually higher than that on capital gains. Thus $\tau_{ct} \geq \tau_{gt}$.

We now modify the short-selling conditions in Assumption 6 as follows.

Assumption 8. *When investors short sell an asset i in the taxable account, they are required to pay coupon payments with a full tax deduction at the rate τ_{ct}. The capital gains and losses on short positions are realized annually and taxed at the rate τ_{gt}.*

Under these additional assumptions, the optimization problem is still described by Equations (2) and (3), with after-tax returns in Equation (11). The following lemma shows that the replication argument in Lemma 1 continues to hold.

Lemma 3. *Under Assumptions 2–5 and 7–8, Lemma 1 remains valid with the following single-period replication portfolio (x_{it}, y_{it}) and replication cost z_{it}:*

$$x_{it} = \frac{1}{1 - \tau_{gt}}, \quad y_{it} = \frac{1}{R_{ft}} \left(\frac{c_{it}(\tau_{ct} - \tau_{gt})}{1 - \tau_{gt}} - \frac{\tau_{gt}}{1 - \tau_{gt}} \right), \quad \textit{and } z_{it} = x_{it} + y_{it}. \tag{12}$$

Like Lemma 1, this lemma can be proved by combining Equations (11) and (12) to show that the after-tax return of the taxable portfolio (x_{it}, y_{it}), $x_{it}\,\widetilde{R}_{it} + y_{it}\,R_{ft}$, is equal to $\tilde{r}_{it}$, the return of asset i in the tax-deferred account. The lemma states that each tax-deferred dollar invested in asset i is equivalent to z_{it} dollars in

[14] Although c_{it} may be stochastic due to changing prices in the future, it is reasonable to assume that it is deterministic over the next period, since both C_{it} and P_{it} are known at t. Using arguments similar to those in Section 4.2, we can show that the location result in this section goes through, provided that c_{it} is deterministic over the next period, even if it may change stochastically in the future. We ignore this possibility in the remainder of the paper for simplicity.

the taxable account. Thus, the value z_{it} captures the tax efficiency of asset i. To be consistent with conventional notions of tax efficiency, which is usually expressed in terms of tax rates of assets, we introduce the concept of *effective tax rate* in the following definition.

Definition 2. Let z_{it} be the single-period replication cost of asset i. We define $\hat{\tau}_{it}$ as the *effective tax rate* of asset i if a hypothetical asset whose return is paid out in full and taxed at the rate $\hat{\tau}_{it}$ has the same replication cost z_{it}. In particular, $\hat{\tau}_{it}$ solves

$$\frac{1}{1-\hat{\tau}_{it}} - \frac{\hat{\tau}_{it}}{(1-\hat{\tau}_{it})R_{ft}} = z_{it}. \tag{13}$$

Since the hypothetical asset is of the type that we consider in the benchmark case in which all returns of the asset are paid out in full and taxed uniformly at the asset-specific rate, the left-hand side of Equation (13), which is derived by replacing the tax rate τ_{it} with $\hat{\tau}_{it}$ in Equation (5), calculates the replication cost for the hypothetical asset. The right-hand side of Equation (13) is the single-period replication cost based on the realistic tax code.

Following this definition, we connect the more realistic tax code to our benchmark case by defining an asset-specific tax rate for each asset based on its tax efficiency. In particular, the effective tax rate provides a single measure that embodies the effect of differential tax rates on capital gains and ordinary income, and the percentage of income taxed as ordinary income and capital gains. The following corollary illustrates this point.

Corollary 1. *Under Assumptions 2–5 and 7–8, the effective tax rate in Definition 2 equals*

$$\hat{\tau}_{it} = \omega_{ct}\,\tau_{ct} + (1-\omega_{ct})\tau_{gt}, \quad \textit{where}$$
$$\omega_{ct} = \frac{(1-\tau_{gt})c_{it}}{(1-\tau_{gt})c_{it} + (1-\tau_{ct})(r_{ft}-1-c_{it})}. \tag{14}$$

The corollary is derived by substituting the replication cost z_{it} in Equation (12) into the definition of $\hat{\tau}_{it}$ in Equation (13). The effective tax rate can be expressed as a weighted average of the tax rates on the different return components, τ_{ct} and τ_{gt}, with weights depending only on the coupon rate for the asset c_{it} and the risk-free rate r_{ft}. Neither the capital gains distribution $\tilde{g}_{it}$ nor individual preferences affect the effective tax rate. The reason goes back to the replication argument, which clearly is preference-free and independent of the expected return of assets. Since all assets should have the same return on a risk-adjusted basis to preclude arbitrage, for an asset with a coupon rate of c_{it}, the effective risk-adjusted capital gains rate has to be $r_{ft} - 1 - c_{it}$. Instead of $\tilde{g}_{it}$, this risk-adjusted rate enters the denominator of ω_{ct} and determines the effective tax rate $\hat{\tau}_{it}$.

The fact that the effective tax rate does not directly depend on $\tilde{g}_{it}$ drastically simplifies the process of ranking assets by their tax rates. We can show that $\hat{\tau}_{it}$ is monotonically increasing in the coupon rate c_{it} as long as $\tau_{ct} > \tau_{gt}$. In particular, when $c_{it} = 0$, the weight ω_{ct} is zero, and $\hat{\tau}_{it} = \tau_{gt}$. When $c_{it} = r_{ft} - 1$, the weight ω_{ct} equals 1, and $\hat{\tau}_{it} = \tau_{ct}$. In general, the higher the coupon rate (or dividend payments for stocks), the higher the effective tax rate.

One might find it surprising that according to Equation (14) it is possible to have $\hat{\tau}_{it} > \tau_{ct}$ for assets with a coupon rate higher than the risk-free rate (which is quite likely, for example, for long-term bonds). The reason is that, if an asset pays a safe coupon rate c_{it} that is higher than the risk-free rate, then on average investors should expect a capital loss to preclude arbitrage. According to Assumption 8, capital losses are taxed at the lower rate τ_{gt}. Hence, investors are expected to pay the high rate τ_{ct} on the coupon payments while they receive low tax credits on the expected capital losses, leading to a very high effective tax rate. In reality, investors can optimally realize these capital losses as short-term losses. As long as they can use them to offset short-term capital gains or income, the effective tax credit on losses is higher (usually the same as τ_{ct}), and the effective tax rate on this asset is lower (also close to τ_{ct}). For simplicity, we do not consider the different tax rates on short-term versus long-term capital gains.[15]

Given the ranking of assets by the effective tax rate, we can extend the benchmark results to the current setting by utilizing the notion of effective tax rates and the replication portfolios in Lemma 3. The reason is that Propositions 1 and 2 rely exclusively on the replication cost z_{it}, and that $\hat{\tau}_{it}$ in Equation (13) is a monotonically increasing function of z_{it}.[16] The following proposition summarizes both the location and the allocation results.

Proposition 3. *Under Assumptions* 2–5 *and* 7–8, *Propositions* 1 *and* 2 *hold with* $\hat{\tau}_{it}$, *as defined in Equation* (14), *in place of* τ_{it}, *and with* x_{it}, y_{it}, *and* z_{it} *as defined in Equation* (12).

The proof of the proposition is straightforward, since the proof of Proposition 1 relies exclusively on the replication cost of assets, which is fully captured by our definition of $\hat{\tau}_{it}$. As to the allocation result in Proposition 2, although the replication portfolio in Equation (12) differs from that in Equation (5), the proof goes through once we substitute in the new replication portfolio. Thus,

15 Constantinides (1984); and Constantinides and Ingersoll (1984) provide detailed analyses of differential long-term and short-term capital gains rates for stocks and bonds when investors have access to only a taxable account.

16 This is true as long as $z_{it} > 1$. For assets with $z_{it} < 1$, investors are worse off placing them in the tax-deferred account, and hence these assets should never be placed in the tax-deferred account.

once we properly define the effective tax rate $\hat{\tau}_{it}$ for each asset, both location and allocation results in the benchmark case remain valid.

In particular, since higher coupon bonds or higher-dividend stocks have higher effective tax rates according to Corollary 1, investors prefer to place these assets in the tax-deferred account. Note that market completeness with respect to price risk is still not required for either the replication argument or the location and allocation results.

3.2 With a tax-timing option on capital gains

We now consider the case in which risky asset returns give investors the tax-timing option of postponing the realization of capital gains and accelerating the realization of tax losses. The following assumption specifies the tax environment.

Assumption 9. *The before-tax return $\tilde{r}_{it}$ on any asset i and the tax rate τ_{ct} on coupon payments are exactly the same as those in Assumption 7. Capital gains and losses are taxed only upon realization at the rate τ_{gt}.*

Thus, instead of paying capital gains taxes annually upon accrual as in Assumptions 1 and 7, under Assumption 9 investors can postpone paying taxes until they realize the capital gain. This assumption is closer to reality. However, for tractability we still leave out some details of the tax code. For example, we allow investors to receive tax credits on losses immediately[17] and still ignore the differential tax rates on short- and long-term capital gains.

With the option to defer the realization of capital gains, it is also necessary to specify the tax treatment on the terminal date.

Assumption 10. *On the terminal date T, the unrealized capital gains can be handled in one of the two ways: (i) if T corresponds to the end of his life span, the investor can escape capital gains taxes through "step-up in basis," or (ii) if T corresponds to a liquidation of the asset for any other reason, the investor is required to realize the capital gains.*

The step-up in basis assumption corresponds to a special provision in the current tax code that allows investors to escape capital gains taxes upon their death,[18] making stocks more attractive in taxable accounts, especially for older

[17] In reality, investors are only allowed to realize tax credits on losses through offsetting their realized capital gains, dividends, or up to $3,000 of their ordinary income, with any extra losses carried over to future years. Moreover, they may be constrained by the "wash sale" rule that restricts the realization of capital losses on assets if they purchase the same asset within thirty days before or after the sale (i.e., a 60-day window).

[18] Specifically, they can pass the taxable assets with embedded capital gains to their estates, and their beneficiaries can avoid paying capital gains taxes by keeping the assets and recording the current market prices as their tax bases for the assets.

investors. The assumption of the required realization of capital gains approximates the tax treatment when investors sell assets for any other reason. We consider both tax treatments in this section.

If investors can choose when to realize capital gains, the tax liability depends nonlinearly on past returns. The before-tax returns can no longer be spanned by the after-tax returns. It is important that the market be complete in this, case so that a dynamic portfolio of asset i and the risk-free bond is sufficient to span the return space generated by any position in the asset. We introduce the following assumption that effectively completes the market.[19]

Assumption 11. *The price of risky asset i follows a binomial process from t to* $t+1$,

$$\widetilde{P}_{it+1} = \begin{cases} u_{it} P_{it}, \text{ with probability } p_i, \\ u_{it}^{-1} P_{it}, \text{ with probability } 1 - p_i, \end{cases} \quad \text{where } u_{it} > 1, i = 2, \ldots, N. \tag{15}$$

Hence, the before-tax return $\tilde{r}_{it}$ *is also binomial and equals* r_{it}^H *and* r_{it}^L, *respectively, where*

$$r_{it}^H = c_{it} + u_{it}, \qquad r_{it}^L = c_{it} + u_{it}^{-1}. \tag{16}$$

The market is effectively complete in the sense that a martingale pricing measure can be defined, even though dividend and capital gains are taxed differently and the same asset receives different tax treatment if placed in taxable or tax-deferred accounts (Ross, 1987).

We further modify the borrowing and short-selling conditions in Assumption 8 to rule out arbitrage opportunities (see Constantinides (1983) for a detailed discussion).

Assumption 12. *When investors short sell an asset i in the taxable account at time t, they have full use of the proceeds* P_{it} *and pay annual coupon payments with a full tax deduction. They are required to "mark-to-market" by posting* P_{is} *dollars as cash collateral at any time* $s \geq t$ *until they close the short position. The capital gains and losses on the short position are taxed only upon realization at the rate* τ_{gs}. *They earn a "rebate interest rate"* $\widehat{r}_{is}$ *on the cash collateral, which is specified in Equation (A3f) of Appendix A to rule out arbitrage.*

Full tax deduction on coupon payments is reasonably close to reality. The use of short sale proceeds, however, is rather complicated in reality, and typically institution-specific. In general, investors are required to mark-to-market in the

[19] As long as the market is complete, the assumption of a binomial return distribution is not essential. Constantinides (1983) derives the same result in continuous time for returns that follow Wiener processes.

sense that the required amount of cash collateral moves with market prices. The rebate interest rate is typically lower than the risk-free rate and is *exogenously* determined based on investor characteristics and market conditions.[20] Following Constantinides (1983), in this paper we assume that $\widehat{r}_{it}$ is defined *endogenously* to rule out arbitrage opportunities.

Under these new assumptions, W_t^T is no longer sufficient to describe the taxable positions. Instead, the different purchase prices (i.e., cost bases) of all shares for the same asset become state variables. For any investor, let $\widehat{P}_{it} = (P_{i0}, \ldots, P_{it}, 0, \ldots, 0)'$ be the vector of his potential cost bases and $h_{it} = (h_{it0}, \ldots, h_{itt}, 0, \ldots, 0)'$ be the vector of his share holdings of asset i at time t, with h_{its} being the number of shares purchased at time s with a cost basis of P_{is}. The zeros at the end are added to ensure that both $\widehat{P}_{it}$ and h_{it} have a length of T. Let $H_t = (h_{1t}, \ldots, h_{Nt})$ be the vector of all holdings and $\widehat{c}_{is} = (1-\tau_{cs})\, c_{is} P_{is}$ be the after-tax coupon payment. Instead of Equations (2)–(3), the dynamic optimization problem can be written as

$$J_t\left(H_{t-1}, \widehat{P}_{it}, W_t^D\right) = \max_{\{H_s,\, \Theta_s^D\}_{s=t}^{T-1}} \mathrm{E}_t\left[u\left(W_T^T + W_T^D\right)\right], \tag{17}$$

subject to Equations (3b) and (3c), and the following budget constraint in the taxable account:

$$\sum_{i=1}^{N} \left[\widehat{c}_{is}\, \mathbf{1}_T' h_{is-1} + \left((1-\tau_{gs})P_{is}\, \mathbf{1}_T' + \tau_{gs}\widehat{P}_{is}'\right)(h_{is-1} - h_{is})\right] = 0,$$

$$s = t, \ldots, T-1, \tag{18a}$$

$$W_T^T = \begin{cases} \sum_{i=1}^{N} \left[(\widehat{c}_{iT} + P_{iT})\, \mathbf{1}_T' h_{iT-1}\right] & \text{if escape terminal taxes,} \\ \sum_{i=1}^{N} \left[(\widehat{c}_{iT} + (1-\tau_{gT})P_{iT})\, \mathbf{1}_T' h_{iT-1} \right. & \\ \left. +\tau_{gT}\widehat{P}_{iT}' h_{iT-1}\right] & \text{if realize terminal gains.} \end{cases} \tag{18b}$$

Clearly, the terminal wealth W_T^T depends on the terminal tax treatment in Assumption 10.

Constantinides (1983, 1984) shows that in a complete market, the trading strategy on any given share is separable from the overall portfolio choices when investors have access to only a taxable account. He derives both the optimal trading strategy and the closed-form expression for the value of a long and a short position with embedded capital gains. For completeness, we reproduce Theorem 1 of Constantinides (1983) in the following lemma.

[20] Institutional investors need to post 102% of the market value of the borrowed shares on the transaction date as cash collateral, earning a rebate interest rate on the collateral as determined by market conditions, such as the volatility of the asset and the difficulty of locating the shares to be shorted (see for example, Duffie, Gârleanu, and Pedersen, (2002).) Retail investors, on the other hand, may be required to post as much as 150% of the market value as cash collateral and typically do not receive interest payments on their cash collateral.

Lemma 4 (Constantinides, 1983). *Under Assumptions* 2–5 *and* 9–12*, the optimal trading strategy on any taxable position is to realize losses immediately and to postpone gains until the terminal date.*

The optimality of postponing gains and realizing losses relies on the fact that investors can always introduce new holdings in the asset in order to achieve any preferred risk profile. For example, if there are losses on a long position and the optimal portfolio requires retaining the long position, investors do strictly better by realizing the losses immediately to get tax credits and then repurchasing the same asset, because they can earn interest on the tax credits received. Similarly, if there are capital gains on the long position and investors need to reduce the total risk exposure to the asset, instead of directly liquidating their long position, investors can improve their utility by retaining the share while simultaneously shorting the same asset. Again, the benefits are the interest payments saved on capital gains.

Theorem 2 of Constantinides (1983) and Proposition 2 of Constantinides (1984) show that when the investment horizon goes to infinity, any taxable position with an embedded capital gain is equivalent to a cash position in the taxable account invested in a portfolio of the investor's choice. Constantinides also derives a closed-form expression for the equivalent cash value. In the following lemma, we extend Constantinides' results to consider the case in which investors have a finite horizon T. Since the benefit of tax-deferred accounts changes significantly as the horizon changes, the assumption of a finite horizon is crucial to our results. However, this assumption makes terminal tax treatment relevant and drastically complicates the calculation. We are no longer able to derive a closed-form expression for the equivalent cash value, but instead can define it recursively.

Lemma 5. *Under Assumptions* 2–5 *and* 9–12*, investors following the optimal strategy in Lemma* 4 *are indifferent between having a share of asset* i, $i = 2, \ldots, N$, *with price* P_{it} *and cost basis (i.e., the purchase price)* $\widehat{P}_i$, *and having full use of* $V_{it}(P_{it}, \widehat{P}_i)$ *dollars of cash in the taxable account. Moreover, the value* $V_{it}(P_{it}, \widehat{P}_i)$ *can be calculated recursively.*

(i) *On the terminal date* T, *depending on the terminal tax treatment,*

$$V_{iT}(P_{iT}, \widehat{P}_i) = \begin{cases} P_{iT}, & \text{if } P_{iT} \geq \widehat{P}_i \text{ and escape terminal taxes,} \\ P_{iT} - \tau_{gT}(P_{iT} - \widehat{P}_i), & \text{if } P_{iT} < \widehat{P}_i \text{ or if realize terminal gains.} \end{cases} \tag{19}$$

(ii) *At any time* $t < T$, *given* $V_{it+1}(\widetilde{P}_{it+1}, \widehat{P}_i)$, *we have*

$$V_{it}(P_{it}, \widehat{P}_i) = \begin{cases} x_{it}^T + y_{it}^T, & \text{if } P_{it} \geq \widehat{P}_i, \\ P_{it} - \tau_{gt}(P_{it} - \widehat{P}_i), & \text{if } P_{it} < \widehat{P}_i, \end{cases} \tag{20}$$

where

$$x_{it}^{T}(P_{it},\widehat{P}_i)=P_{it}\,\frac{R_{it}^{H}(\widehat{P}_i)-R_{it}^{L}(\widehat{P}_i)}{R_{it}^{H}(P_{it})-R_{it}^{L}(P_{it})},\tag{21a}$$

$$y_{it}^{T}(P_{it},\widehat{P}_i)=\frac{P_{it}}{R_{ft}}\,\frac{R_{it}^{H}(P_{it})R_{it}^{L}(\widehat{P}_i)-R_{it}^{L}(P_{it})R_{it}^{H}(\widehat{P}_i)}{R_{it}^{H}(P_{it})-R_{it}^{L}(P_{it})},\tag{21b}$$

and

$$R_{it}^{H}(\widehat{P}_i)=(1-\tau_{ct})\,c_{it}+\frac{1}{P_{it}}\,V_{it+1}(u_{it}P_{it},\widehat{P}_i),\tag{22a}$$

$$R_{it}^{L}(\widehat{P}_i)=(1-\tau_{ct})\,c_{it}+\frac{1}{P_{it}}\,V_{it+1}(u_{it}^{-1}P_{it},\widehat{P}_i).\tag{22b}$$

On the terminal date, if $P_{iT}\geq\widehat{P}_i$ and investors can escape taxes through step-up in basis, each share is worth P_{iT} dollars and $V_{iT}=P_{iT}$. If, instead, investors are required to realize capital gains or if $P_{iT}<\widehat{P}_i$ so that investors optimally choose to realize losses, then Equation (19) describes the after-tax value of the taxable position.

At any time t before the terminal date, if $P_{it}<\widehat{P}_i$, investors optimally liquidate the position to realize capital losses. Under Assumption 9, they receive tax credits immediately. Thus, the value of the position is equal to the after-tax proceeds described in Equation (20). If $P_{it}>\widehat{P}_i$, investors optimally hold the position for another period. Each dollar of this position yields an after-tax coupon of $(1-\tau_{ct})\,c_{it}$ dollars and $1/P_{it}$ shares of asset i with basis $\widehat{P}_i$, which is worth $V_{it+1}(\widetilde{P}_{it+1},\widehat{P}_i)$ per share by the induction assumption. Thus, $\widetilde{R}_{it}(\widehat{P}_i)$ in Equation (22) gives the effective after-tax return on the position of asset i with cost basis $\widehat{P}_i$. Similarly, using Equations (21)–(22), we can verify that a portfolio (x_{it}^{T},y_{it}^{T}) of newly purchased asset i and bonds has an effective after-tax return of $x_{it}^{T}\widetilde{R}_{it}(P_{it})+y_{it}^{T}R_{ft}$, which is equal to $\widetilde{R}_{it}(\widehat{P}_i)$. Thus, the cash value of the position $V_{it}(P_{it},\widehat{P}_i)$ is equal to the cost of the portfolio, $x_{it}^{T}+y_{it}^{T}$.

While the above replication is conceptually simple, its implementation requires netting out long and short positions dynamically. In particular, to effectively make full use of the $V_{it}(\cdot)$ dollars, the investor needs to borrow y_{it}^{T} dollars and short sell x_{it}^{T} dollars of asset i. Under Assumption 12, the investor has full use of the V_{it} dollars. However, in addition to this cash position, the investor has a portfolio that is (i) long one share of asset i with basis $\widehat{P}_i$, (ii) short x_{it}^{T} dollars of asset i with basis P_{it}, and (iii) short y_{it}^{T} dollars of bonds paying interest rate R_{ft}. Although (as we show in the appendix) the effective risk exposure of portfolio (i)–(iii) is zero at time t, the positions will further evolve and may remain open until the terminal date. Specifically, it is necessary that the investor follows the optimal strategy of realizing losses and postponing gains for both the long position with basis $\widehat{P}_i$ in part (i) and the short position with basis P_{it} in part (ii) for all future dates.[21] Hence, keeping track of all the open positions is extremely complex. Fortunately, for our purpose of deriving the optimal portfolio strategy in both accounts, it is okay to ignore portfolio (i)–(iii) and all the corresponding future positions, since their future risk exposure always cancels each other.

Lemma 5 also allows us to define the effective single-period after-tax return $\widetilde{R}_{it}$ on a newly purchased share of asset i as $R^H_{it}(P_{it})$ and $R^L_{it}(P_{it})$ using Equation (22). Like the cash value V_{it}, those returns can only be defined recursively, assuming that investors follow the optimal trading strategy in Lemma 4 on all future dates. Nonetheless, these returns are realizable by the following well-defined strategies. Investors can treat them the same way they do the single-period returns in Equations (1) and (11).

We are now ready to extend the results in the benchmark case to the current setting.

Lemma 6. *Under Assumptions* 2–5 *and* 9–12*, the single-period replication result in Lemma 1 holds trivially for asset* $i = 1$ *with* $\tau_{it} = \tau_{ct}$*. For assets* $i \geq 2$*, as long as investors follow the optimal strategy in Lemma* 4*, the replication result in Lemma* 1 *holds with*

$$x_{it} = \frac{r^H_{it} - r^L_{it}}{R^H_{it}(P_{it}) - R^L_{it}(P_{it})}, \quad y_{it} = \frac{1}{R_{ft}} \frac{R^H_{it}(P_{it}) r^L_{it} - R^L_{it}(P_{it}) r^H_{it}}{R^H_{it}(P_{it}) - R^L_{it}(P_{it})},$$
$$\text{and } z_{it} = x_{it} + y_{it}, \tag{23}$$

where r^H_{it} *and* r^L_{it} *are defined in Equation (16) and* R^H_{it} *and* R^L_{it} *in Equation (22).*

Like Lemmata 1 and 3, this lemma can be proved by combining Equations (22) and (23) to show that the effective after-tax return on the portfolio (x_{it}, y_{it}) of newly purchased asset i and risk-free bonds is equal to $\tilde{r}_{it}$, the single-period payoff of a tax-deferred dollar invested in asset i. The value z_{it} again represents the single-period replication cost. Like the properties of V_{it} and $\widetilde{R}_{it}$, although described as a single-period result, the above replication is implemented dynamically in the sense that investors need to follow Lemma 4 on all future positions. Lemmata 5 and 6 ensure that it is possible for investors with a taxable portfolio (x_{it}, y_{it}) to walk away with $\tilde{r}_{it}$ dollars in cash at $t + 1$ and have no further risk exposure. Therefore, for the purposes of illustrating the overall risk exposure, we can treat x_{it}, y_{it}, and z_{it} as if they were realized over a single period. The following proposition extends the location and allocation results of the benchmark case.

[21] For example, if $\widetilde{P}_{it+1} > P_{it} > \widehat{P}_i$, then the long position still has a gain, and the short position has a loss. The investor optimally closes the short position with basis P_{it}, while he maintains the long position with basis $\widehat{P}_i$. In addition, to make full use of the $V_{it+1}(\widetilde{P}_{it+1}, \widehat{P}_i)$ dollars in cash for the long position, he needs to establish a new short position with a basis $\widetilde{P}_{it+1}$, which he maintains until time $t + 2$.

Proposition 4. *Under Assumptions* 2–5 *and* 9–12*, let* $\hat{\tau}_{it}$ *be the effective tax rate in Definition* 2 *for* z_{it} *defined in Equation* (23)*. Then:* (i) *The optimal tax-deferred portfolio* $\Theta_t^D = (\theta_{1t}^D, \ldots, \theta_{Nt}^D)'$ *has the same form as Equation* (7) *wth* $\hat{\tau}_{it}$ *in place of* τ_{it}*.* (ii) *For any investor following the optimal tax-deferred strategy, let* Z_t *be the equivalent taxable wealth in Equation* (8) *with* z_{it} *in Equation* (23)*, then*

$$J_t\big(H_{t-1}, \widehat{P}_{it}, W_t^D\big) = J_t(\widehat{H}_{t-1}, \widehat{P}_{it}, 0), \quad \textit{where } \widehat{H}_{t-1} = H_{t-1} + Z_t W_t^D \boldsymbol{e}_{TN}^{t\,1}, \quad (24)$$

$J_t(\cdot)$ *is as defined in Equation (17), and* $\boldsymbol{e}_{TN}^{ij}$ *is an index array of size* $T \times N$ *that equals 1 for the* ij*th element and 0 otherwise. (iii) Let* $H_t^*(H_{t-1}, \widehat{P}_{it})$ *be the optimal portfolio when the investor has access to only a taxable account, then the optimal taxable portfolio is*

$$H_t\big(H_{t-1}, \widehat{P}_{it}, W_t^D\big) \equiv H_t^*(\widehat{H}_{t-1}, \widehat{P}_{it}, 0) - \big(x_{1t}\theta_{1t}^D + y_t'\Theta_t^D\big) Z_{t+1} W_t^D \boldsymbol{e}_{TN}^{t\,1} - \sum_{i=2}^{N} x_{it}\theta_{it}^D Z_{t+1} W_t^D \boldsymbol{e}_{TN}^{t\,i},$$

where $y_t = (y_{1t}, \ldots, y_{Nt})'$*, and both* x_{it} *and* y_{it} *are defined in Equation (23).*

This proposition takes into account the fact that $\{H_{t-1}, \widehat{P}_{it}\}$, rather than W_t^T, is the state variable when investors are allowed to postpone capital gains. Both location and allocation results remain valid, since they depend mainly on the replication cost z_{it}, which is well defined from Lemma 6. In particular, the tax-deferred portfolio is still independent of the taxable portfolio, even though the definition of $\hat{\tau}_{it}$ requires that investors follow the optimal strategy in Lemma 4 for all future periods. Investors still rank assets by their tax efficiency and place only the highest taxed assets in the tax-deferred account. As Equation (24) indicates, the two-account problem can still be mapped to a single-account problem, with a total effective taxable holding of $\widehat{H}_{t-1}$, which is equal to the existing taxable holding plus an additional wealth component of $Z_t W_t^D$. To derive their optimal taxable holding, investors subtract the effective risk exposures of their tax-deferred holdings from the single-account solution $H_t^*(\cdot)$.

It is worth noting that if an investor follows a suboptimal strategy in the taxable account, for example, realizing gains immediately, then the effective tax rate is higher for the asset. However, it would not be optimal to keep that asset in the tax-deferred account, since the investor could be better off by keeping this asset in the taxable account while holding other truly highly taxed assets in the tax-deferred account. Essentially, there are two steps in the optimization. First, following the optimal taxable strategy yields the lowest effective tax rate for any given asset. Second, using these optimal effective tax rates, investors pick the highest taxed asset to place in the tax-deferred account. If an investor follows a suboptimal taxable strategy, he may still maximize the second-step tax benefit, but not the first-step benefit, and will be worse off overall.

3.3 Properties of the effective tax rate

Propositions 3 and 4 suggest that once we properly define the effective tax rate of assets, location and allocation decisions in the presence of a tax-deferred account become straightforward. For practical purposes, therefore, it is important to understand the properties of the effective tax rate $\hat{\tau}_{it}$. We numerically calculate the equivalent taxable wealth z_{it} as defined in Equation (23) and the corresponding effective tax rate in Equation (13).[22] In Figure 1, we plot $\hat{\tau}_{it}$ as a function of the remaining investment horizon $T - t$. The solid lines in all panels of the figure correspond to the case in which investors are required to realize capital gains on the terminal date, while the dashed lines correspond to the case in which investors are allowed to escape terminal capital gains taxes through step-up in basis.

The effective tax rate $\hat{\tau}_{it}$ exhibits several interesting features. First, the effective tax rate decreases with the remaining investment horizon when investors are required to realize terminal capital gains, and increases with the remaining horizon when investors can escape terminal taxes. For a sufficiently long horizon, the impact of terminal tax treatments diminishes, and the solid and dashed lines converge in all panels. For a shorter horizon (say, less than five years), however, the effective tax rate differs significantly for different terminal tax treatments and is much higher when investors are required to realize capital gains. Second, by comparing results within each column, we see that the effective tax rate increases with coupon payments and converges to zero over a long horizon when $c_{it} = 0$ (Panels A and B). Third, by comparing results between the two columns, we see that the effective tax rate is lower for assets with higher volatility.

To understand the horizon effect, consider the case in which investors are required to realize terminal capital gains. Right before the terminal date, i.e., when $T - t = 1$, there is no longer a timing option, and the results are identical to those in Equation (14). In particular, $\hat{\tau}_{it} = 20\%, 33\%$, and 40% when $c_{it} = 0$, 3%, and 5%, respectively. For a longer remaining horizon, investors enjoy the benefit of postponing gains until the terminal date while realizing losses immediately. Hence, the longer the horizon, the more beneficial it is to hold assets in the taxable account, and the lower the effective tax rate $\hat{\tau}_{it}$.

On the other hand, if investors are allowed to escape terminal taxes, the benefit of tax timing is highest right before the terminal date. Investors can realize losses if the price goes up over the period, and they can completely

[22] If $z_{it} < 1/R_{ft} < 1$, it is not beneficial to place asset i in the tax-deferred account, and the effective tax rate $\hat{\tau}_{it}$ should be negative. However, mathematically it is possible to get $\hat{\tau}_{it} > 0$. We set $\hat{\tau}_{it} = -1$ in this case. This assumption does not affect our main results since, as long as investors are following the optimal strategy, they never place any asset with $z_{it} < 1$ in the tax-deferred account. Setting $\hat{\tau}_{it} = -1$ yields a consistent ranking of assets by their tax efficiency.

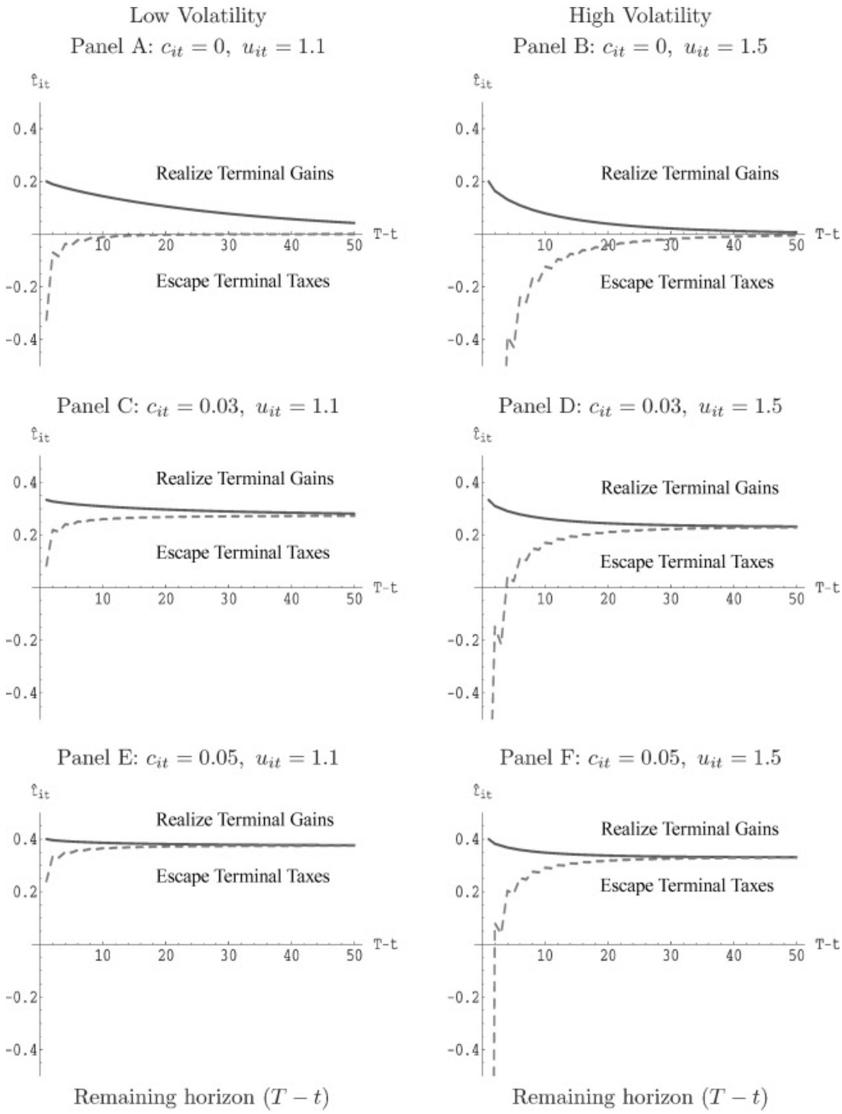

Figure 1

Current-period effective tax rate $\hat{\tau}_{it}$ as a function of the remaining investment horizon $T-t$

In all panels, the solid lines represent the case in which investors are required to realize capital gains on the terminal date and the dashed lines represent the case in which investors are allowed to escape terminal taxes through step-up in basis. In the left column (Panels A, C, and E), we report the case of lower asset volatility by choosing $u_{it} = 1.1$, which implies an asset volatility of 9.5%. In the right column (Panels B, D, and F), $u_{it} = 1.5$ and the asset volatility is 41.7%. The coupon rate is $c_{it} = 0\%$ in Panels A and B, 3% in Panels C and D, and 5% in Panels E and F. Other parameters are set at the following values: $r_{ft} = 5\%$, $\tau_{ct} = 40\%$, and $\tau_{gt} = 20\%$.

escape capital gains taxes if the price goes down. As a result, it is very beneficial to hold risky assets in the taxable account. In particular, the effective tax rate is negative in all panels other than C and E, suggesting that old investors (with a short horizon) may be better off keeping the asset in the taxable account even if they are given the option to place this asset in the tax-deferred account. As the horizon increases, although investors can still postpone capital gains and escape taxation on the terminal date, the benefit is reduced. The reason is that postponing capital gains may reduce the benefit of realizing future losses. For example, at time $T-2$, if the price goes up at $T-1$, investors can postpone the gain until T. They eventually can escape the terminal tax only if the price goes up further at T. If instead the price goes down at T, investors are effectively taxed on the postponed gain because they are no longer able to realize the new loss that occurred between $T-1$ and T. Hence, the longer the horizon, the less beneficial the timing option and the higher the effective tax rate. When the horizon is long enough, for assets with zero coupon payments ($c_{it}=0$), the benefit of postponing capital gains taxes is exactly offset by the cost of not being able to realize capital losses over all future periods, and $\hat{\tau}_{it}=0$.[23]

The second and third results are straightforward. The tax rate on capital gains is usually lower than that on coupon payments, and the tax-timing option only serves to further reduce the effective tax rate on capital gains. Hence, coupon payments are taxed more heavily than capital gains, and higher coupon assets have higher effective tax rates. The volatility of the assets reduces the effective tax rate, since the tax-timing option of postponing gains and realizing losses is more valuable for more volatile assets.

4. Other Extensions of the Model

In the previous section we show that more realistic tax treatments of investment returns do not materially change our main results. In the remainder of the paper, we return to the benchmark model and relax other assumptions in order to understand the robustness of our results.

4.1 Consumption and contribution decisions

We first relax Assumption 3 to allow for both intermediate consumption and the possibility of either contributing to or withdrawing from the tax-deferred account before the terminal date.

Assumption 13. *Investors have a fixed horizon T and care about both intermediate and terminal consumptions. They can contribute a maximum of $\overline{M}_t$*

[23] Readers familiar with Constantinides (1983) may find this result surprising. Constantinides shows that as long as assets are risky, even if investors escape terminal taxes, the effective tax rate on assets is strictly positive and can be quite close to τ_{gt} for high volatility and high-cost basis assets. Our result is consistent with that of Constantinides (1983), except that our effective tax rate in Definition 2 is different from his. Note that the effective tax rate in Constantinides (1983) is defined as the $\widehat{\tau}$ that solves

$$V_{it}(P_{it}, \widehat{P}_i) = (1-\widehat{\tau})P_{it} + \widehat{\tau}\widehat{P}_i.$$

His $\widehat{\tau}$ captures the effective tax rate on embedded capital gains and is a function of the cost basis of the share. Although investors choose to postpone all gains, the fact that they are realizing losses implies a positive effective tax rate on embedded capital gains. In contrast, our effective tax rate captures the tax efficiency of placing a newly purchased share in the tax-deferred account over the next period.

dollars to the tax-deferred account and are allowed to withdraw by paying a proportional penalty η_t *at any time* $t \leq T$.

Note that $\overline{M}_t$ can be negative, in which case investors are forced to withdraw at least $|\overline{M}_t|$ dollars from the tax-deferred account. Investors receive $(1-\eta_t)$ dollars in the taxable account for each dollar withdrawn from the tax-deferred account in year t. In practice, there are different rules about contribution limits and withdrawal penalties for different types of tax-deferred accounts.[24] Assumption 13 is flexible enough to capture most of these rules.

The optimization problem in Equation (2) can be modified as follows:

$$J_t(W_t^T, W_t^D) = \max_{\{C_{s+1}, M_{s+1}, \Theta_s^T, \Theta_s^D\}_{s=t}^{T-1}} \mathrm{E}_t \left[\sum_{s=t}^{T-1} \left[\delta^{s-t}\, u(c_s) \right] + \delta^{T-t}\, U(C_T) \right], \quad (25)$$

where δ is the time discount, $u(\cdot)$ is the utility over consumption, and $U(\cdot)$ is the utility over terminal bequest (which may include terminal consumption). In addition to Equation (3c), we have the following constraints at any time $s = t, \ldots, T-1$:

$$W_{s+1}^T = W_s^T\, \Theta_s^{T\prime} \widetilde{R}_s - C_{s+1} - (1 - \eta_{s+1}\, 1_{\{M_{s+1}<0\}})\, M_{s+1}, \quad (26a)$$

$$W_{s+1}^D = W_s^D\, \Theta_s^{D\prime} \tilde{r}_s + M_{s+1}, \quad (26b)$$

$$M_{s+1} \leq \overline{M}_{s+1}, \quad W_{s+1}^D \geq 0, \quad W_T^T \geq 0, \quad (26c)$$

where $1_{\{M_s<0\}}$ is an indicator function that equals 1 if and only if $M_s < 0$. Equations (26a) and (26b) are the wealth evolutions adjusted for consumption and contribution, and Equation (26c) imposes the contribution limit and the positive wealth constraint. Note that taxable wealth is allowed to be negative before the terminal date, indicating that investors are free to borrow in order to consume or to contribute to their tax-deferred accounts.

Since investors are allowed to borrow in the taxable account, we can show that the optimal contribution decision is to always maximize the contribution in order to maximize the total tax benefit. Moreover, the only impact of tax-deferred accounts on the consumption decision is the wealth effect. Formally, we have the following proposition.

Proposition 5 (consumption and contribution). *Under Assumptions* 1*,* 2*,* 4–6*, and* 13*:* (i) *Investors always contribute the maximum amount allowed to their tax-deferred accounts, or* $M_t = \overline{M}_t$. (ii) *Let* $C_t^*(W_t^T)$ *be the optimal consumption when investors have access to only a taxable account, then the*

[24] For example, the contribution limit for 401(k) plans is the minimum of a fixed percentage of the individual's salary (which usually varies by employer) and a fixed maximum amount set by the government, which is \$15,500 for year 2007. Investors face a 10% penalty if they withdraw from their 401(k) or traditional IRA accounts before the age of $59\frac{1}{2}$, and are also required to take annual minimum distributions from these accounts after the age of 70, based on their average life expectancy. Roth IRAs usually have no required distribution after the age of 70 and no penalty for some qualified early withdrawals.

optimal consumption with a tax-deferred account is

$$C_t^M\left(W_t^T, W_t^D\right) = C_t^*\left(\widehat{W}_t^T\right), \qquad \textit{where } \widehat{W}_t^T = W_t^T + Z_t W_t^D + Z_t^M, \tag{27}$$

Z_t *is the equivalent taxable wealth in Equation* (8)*, and* Z_t^M *is defined recursively as*

$$Z_T^M = 0, \quad \textit{and} \quad Z_t^M = \overline{M}_t(Z_t - 1) + \frac{Z_{t+1}^M}{R_{ft+1}}, \qquad t < T. \tag{28}$$

(iii) *The optimal taxable and tax-deferred portfolios are the same as those in Propositions* 1 *and* 2 *with the above defined effective wealth* $\widehat{W}_t^T$.

The proposition states that the opportunity to make future contributions to the tax-deferred account is equivalent to increasing the endowment of safe assets in the taxable account. Moreover, the optimal tax-deferred portfolio is not affected at all by the contribution or consumption decisions. The optimal taxable portfolio is similar to that derived in Proposition 2 except that investors need to account for an additional wealth adjustment Z_t^M, which is the present value of tax benefits for all future contributions. Investors never voluntarily withdraw from the tax-deferred account before the terminal date even if there is no penalty to withdraw, since they are better off maximizing the tax benefit by contributing as much as possible. They can always borrow in their taxable accounts to meet any liquidity or consumption needs.

4.2 Stochastic tax rates

We now relax the assumption of a deterministic tax rate τ_{it}.

Assumption 14. *Let the before- and after-tax returns,* $\tilde{r}_{it}$ *and* $\widetilde{R}_{it}$*, be identical to those in Assumption* 1*. The tax rate* τ_{it} *for return* $\tilde{r}_{it}$ *is known at* t *but is stochastic over time, and* $\tilde{\tau}_{it+1}$ *is independent of* $\tilde{r}_{it}$.

We assume that the tax rate is deterministic over the next period, even though it may change for future periods. This assumption captures the main impact of stochastic tax rates while avoiding the complexity of having two underlying state variables. It is also realistic considering the usual time lag between the proposal of a tax change and the eventual implementation of the new tax code. We still focus on Roth accounts for simplicity, although a Roth account is no longer observationally equivalent to a 401(k) account due to the uncertain terminal tax rate.[25]

Under the new assumption, the optimization problem is still specified by Equations (2) and (3). Since τ_{it} is known, the single-period replication result in Lemma 1 remains valid. However, the dynamic replication in Lemma 2 needs

[25] We can show that the location result in Proposition 6 remains valid for 401(k) accounts.

to be modified, as the equivalent taxable wealth at time $t+1$ depends on future tax rates and is no longer state-independent. The following proposition shows that both the separability of location and allocation decisions and the optimal location strategy of placing highly taxed assets in the tax-deferred account continue to hold.

Proposition 6. *Under Assumptions* 2–6 *and* 14, (i) *the optimal tax-deferred portfolio is identical to that of Equation* (7), *and* (ii) *for any investor following the optimal tax-deferred strategy, there exists* $\widetilde{Z}_{t+1}$ *such that*

$$J_t\big(W_t^T, W_t^D\big) = \mathrm{E}_{Z_{t+1}}\big[J_t\big(W_t^T + \hat{z}_t\,\widetilde{Z}_{t+1}\,W_t^D,\ 0\big)\big], \tag{29}$$

where $\hat{z}_t$ *is the maximum current-period replication cost in Equation* (8).

The proposition shows that although future tax rates may change, the tax-deferred strategy remains myopic each period. Investors rank all assets by their current-period tax rates and place only those assets with the highest tax rate in the tax-deferred account. The reason is that investors can dynamically adjust their portfolios as the tax environment changes in the future. As a result, the changing future tax rate affects current-period decisions only through the term $\widetilde{Z}_{t+1}$. We show in Appendix A that as long as τ_{it} is constant over the next period and that investors are following the optimal tax-deferred strategy in the future, investors can separate the current-period tax-deferred decision from future uncertainties. On the other hand, although Equation (29) maps the two-account problem to a single-account problem, the dependence of $\widetilde{Z}_{t+1}$ on future tax rates introduces a new state variable into the problem. We can no longer derive the optimal taxable portfolio using a preference-free transformation of the optimal taxable-account-only portfolio.

5. Conclusion

We show that under certain conditions the optimal taxable and tax-deferred portfolio decisions are separable. The right to invest in the tax-deferred account is equivalent to receiving a tax subsidy from the government, which depends only on the tax efficiency of assets placed in the tax-deferred account. The optimal tax-deferred portfolio can be determined myopically and consists only of assets with the highest effective tax rate. Therefore, the tax-deferred portfolio fully specifies investors' preferences for asset locations. The overall allocation can be determined by mapping the two-account problem to a taxable-account-only problem with an increased wealth level.

A main contribution of this paper is to lay out all of the assumptions underlying the location and allocation results. It is important to explore the optimal portfolio decisions when some of these assumptions are relaxed. For example, borrowing and short selling is usually limited and costly in reality. Garlappi and Huang (2006) show that if investors are not allowed to borrow in their taxable accounts, the taxable and tax-deferred portfolios are no longer separable, and investors may prefer lower taxed assets in the tax-deferred account. The main intuition is that the tax benefit becomes state-dependent whenever the constraint binds. In addition to maximizing the expected tax benefit (which is the main result in this paper), investors also worry about reducing the volatility of the tax benefit, leading to a possible preference for lower taxed assets in the tax-deferred account. In future research, it would be interesting to relax other assumptions to better understand the optimal portfolio choices between the two accounts, for example, when interest rates or dividend rates are stochastic over time, or when tax rates are stochastic and correlated with asset returns.

Appendix A: Proofs

Proof of Lemma 2

Conditional on holding Θ_t^D, the tax-deferred wealth at time $t+1$ is $W_{t+1}^D = \tilde{r}_t' \Theta_t^D W_t^D$. Let Θ_t^{T*} be the optimal taxable portfolio for an investor with (W_t^T, W_t^D) and Θ_t^D, then

$$J_t(W_t^T, W_t^D \mid \Theta_t^D) \equiv \max_{\Theta_t^T} \mathrm{E} J_{t+1}(W_{t+1}^T, W_{t+1}^D \mid \Theta_t^D), \tag{A1a}$$

$$= \mathrm{E} J_{t+1}(\widetilde{R}_t' \Theta_t^{T*} W_t^T, \tilde{r}_t' \Theta_t^D W_t^D), \tag{A1b}$$

$$= \mathrm{E} J_{t+1}(\widetilde{R}_t' \Theta_t^{T*} W_t^T + Z_{t+1} \tilde{r}_t' \Theta_t^D W_t^D, 0), \tag{A1c}$$

$$= \mathrm{E} J_{t+1}(\widetilde{R}_t' \widehat{\Theta_t^T} \widehat{W}_t^T, 0), \quad \widehat{\Theta_t^T} \equiv (\widehat{\theta_{1t}^T}, \ldots, \widehat{\theta_{Nt}^T}), \tag{A1d}$$

$$\le \max_{\Theta_t^T} \mathrm{E} J_{t+1}(\widetilde{R}_t' \Theta_t^T \widehat{W}_t^T, 0) \equiv J_t(\widehat{W}_t^T, 0), \tag{A1e}$$

where $\widehat{\theta_{1t}^T} = (\theta_{1t}^{T*} W_t^T + (x_{1t}\theta_{1t}^D + y_t'\Theta_t^D) Z_{t+1} W_t^D)/\widehat{W}_t^T$, $\widehat{\theta_{it}^T} = (\theta_{it}^{T*} W_t^T + x_{it}\theta_{it}^D Z_{t+1} W_t^D)/\widehat{W}_t^T$ for $i > 2$, $\widehat{W}_t^T$ is defined in Lemma 2, x_{it} and y_{it} are defined in Equation (5), and $y_t = (y_{1t}, \ldots, y_{Nt})'$. Equations (A1a) and (A1e) are the definitions of value function J_t; equation (A1c) is due to Definition 1 and the state-independency of Z_{t+1}. To derive Equation (A1d), we replace each $\tilde{r}_{it}$ in $\tilde{r}_t = (\tilde{r}_{1t}, \ldots, \tilde{r}_{Nt})'$ by the return of the replication portfolio $x_{it}\widetilde{R}_{it} + y_{it}R_{ft}$ (from Lemma 1). Hence, $\tilde{r}_t'\Theta_t^D$ is equivalent to the return of all replication portfolios $((x_{it}\theta_{it}^D, y_{it}\theta_{it}^D)$ for asset i), with a total cost of $z_t'\Theta_t^D$. Plugging in the definition of $\widehat{W}_t^T$ in Equation (6) and the above $\widehat{\Theta_t^T}$, we can verify that $\widehat{\Theta_t^T}$ is a feasible portfolio (since $\widehat{\Theta_t^T}'\mathbf{1}_N = 1$) and that Equation (A1c) equals Equation (A1d).

To close the proof, we need to show that $J_t(\widehat{W}_t^T, 0) \le J_t(W_t^T, W_t^D \mid \Theta_t^D)$. Similarly, let $\widehat{\Theta_t^T}^*$ be the optimal taxable portfolio for an investor with $(\widehat{W}_t^T, 0)$. Then we have

$$J_t(\widehat{W}_t^T, 0) \equiv \max_{\Theta_t^T} \mathrm{E} J_{t+1}(\widetilde{R}_t' \Theta_t^T \widehat{W}_t^T, 0), \tag{A2a}$$

$$= \mathrm{E} J_{t+1}(\widetilde{R}_t' \widehat{\Theta_t^T}^* \widehat{W}_t^T, 0), \tag{A2b}$$

$$= \mathrm{E} J_{t+1}(\widetilde{R}_t' \widetilde{\Theta_t^T} W_t^T + Z_{t+1} \tilde{r}_t' \Theta_t^D W_t^D, 0), \quad \widetilde{\Theta_t^T} \equiv (\widetilde{\theta_{1t}^T}, \ldots, \widetilde{\theta_{Nt}^T}), \tag{A2c}$$

$$= \mathrm{E} J_{t+1}(\widetilde{R}_t' \widetilde{\Theta_t^T} W_t^T, \tilde{r}_t' \Theta_t^D W_t^D), \tag{A2d}$$

$$\le \max_{\Theta_t^T} \mathrm{E} J_{t+1}(W_{t+1}^T, W_{t+1}^D \mid \Theta_t^D) \equiv J_t(W_t^T, W_t^D \mid \Theta_t^D), \tag{A2e}$$

where $\widetilde{\theta_{1t}^T} = \frac{1}{W_t^T}(\widehat{\theta_{1t}^T}^* \widehat{W}_t^T - (x_{1t}\theta_{1t}^D + y_t'\Theta_t^D) Z_{t+1} W_t^D)$ and $\widetilde{\theta_{it}^T} = \frac{1}{W_t^T}(\widehat{\theta_{it}^T}^* \widehat{W}_t^T - x_{it}\theta_{it}^D Z_{t+1} W_t^D)$.

Proof of Proposition 1

We prove the result using backward induction. On the terminal date T, Assumption 3 ensures that $Z_T = 1$, and the proposition holds trivially. Assume by induction that the proposition holds at time $t+1$, that is, Z_{t+1} is state-independent. Then Lemma 2 holds. From Equation (6), we know that J_t monotonically increases in $\widehat{W}_t^T$, which is maximized when $\Theta_t^{D\prime} z_t$ is maximized. Given the linearity of $\Theta_t^{D\prime} z_t$ and the fact that z_{it} increases with τ_{it}, the tax-deferred portfolio Θ_t^D in Equation (7) maximizes the value function. Moreover, the corresponding single-period replication cost is $\Theta_t^{D\prime} z_t = \hat{z}_t$. From Lemma 2, the equivalent taxable wealth at time t is $Z_t = \hat{z}_t Z_{t+1}$, which is clearly state-independent, confirming the induction assumption.

Proof of Proposition 2

Given Lemma 2 and Proposition 1, we can map the two-account problem to a single-account problem with effective wealth $\widehat{W}_t^T = W_t^T + Z_t W_t^D$, with Z_t given in Equation (8). Hence, the optimal single-account portfolio with wealth $\widehat{W}_t^T$ describes the optimal overall risk exposure, yielding the first term in Equation (9). To derive their optimal taxable portfolio, we subtract investors' effective risk exposure from tax-deferred holdings, yielding the second term in Equation (9). Finally, we normalize the dollar holding by their taxable wealth to derive the proper portfolio weights.

Proof of Lemma 5

We prove Lemma 5, together with the following two results, using backward induction: (i) a short position with cost basis $\widehat{P}_i$ is equivalent to $S_{it}(P_{it}, \widehat{P}_i)$ dollars of cash, and (ii) the short-rebate rate $\widehat{r}_{it} \equiv 1 + (\widehat{R}_{it} - 1)/(1 - \tau_{ct})$ in Assumption 12 is uniquely defined, where

$$S_{iT}(P_{iT}, \widehat{P}_i) = -P_{iT} + \tau_{gT}(P_{iT} - \widehat{P}_i), \tag{A3a}$$

$$S_{it}(P_{it}, \widehat{P}_i) = \begin{cases} -x_{-it}^T + y_{-it}^T, & \text{if } t < T \text{ and } P_{it} \le \widehat{P}_i, \\ -P_{it} + \tau_{gt}(P_{it} - \widehat{P}_i), & \text{if } t < T \text{ and } P_{it} > \widehat{P}_i, \end{cases} \tag{A3b}$$

$$x_{-it}^T = P_{it}\,\frac{R_{-it}^H(\widehat{P}_i) - R_{-it}^L(\widehat{P}_i)}{R_{-it}^H(P_{it}) - R_{-it}^L(P_{it})}, \quad y_{-it}^T = \frac{P_{it}}{R_{ft}}\,\frac{R_{-it}^H(P_{it})R_{-it}^L(\widehat{P}_i) - R_{-it}^L(P_{it})R_{-it}^H(\widehat{P}_i)}{R_{-it}^H(P_{it}) - R_{-it}^L(P_{it})}, \tag{A3c}$$

$$R_{-it}^H(\widehat{P}_i) = \widehat{R}_{-it}^H + (\widehat{R}_{it} - R_{ft}), \quad \widehat{R}_{-it}^H = -(1-\tau_{ct})\,c_{it} + S_{it+1}(u_{it}P_{it}, \widehat{P}_i)/P_{it}, \tag{A3d}$$

$$R_{-it}^L(\widehat{P}_i) = \widehat{R}_{-it}^L + (\widehat{R}_{it} - R_{ft}), \quad \widehat{R}_{-it}^L = -(1-\tau_{ct})\,c_{it} + S_{it+1}\big(u_{it}^{-1}P_{it}, \widehat{P}_i\big)/P_{it}, \tag{A3e}$$

$$\widehat{R}_{it} = \Big(\big(R_{it}^L(P_{it}) - R_{ft}\big)\widehat{R}_{-it}^H(P_{it}) - \big(R_{it}^H(P_{it}) - R_{ft}\big)\widehat{R}_{-it}^L(P_{it})\Big)\Big/\big(R_{it}^H(P_{it}) - R_{it}^L(P_{it})\big). \tag{A3f}$$

On the terminal date T, V_{iT} is well defined in Equation (19) for both terminal tax treatments. For S_{iT}, the tax code does not allow investors to pass short positions into their estate. We assume that investors are always required to realize capital gains, yielding Equation (A3a) as the after-tax value of short positions. We now assume that all results hold at time $t+1$ and prove for time t.

Whenever the position has a loss, i.e., when $P_{it} < \widehat{P}_i$ for the long position or $P_{it} > \widehat{P}_i$ for the short position, investors optimally realize losses and receive tax credits immediately. All the results hold trivially.

When the position has a gain, investors optimally hold the position for another period. For a long position with basis $\widehat{P}_i$, in order to receive V_{it} dollars in cash the investor can borrow y_{it}^T dollars and short sell x_{it}^T dollars of asset i. The investor is also exposed to a portfolio that is (i) long one share of asset i with basis $\widehat{P}_i$, (ii) short x_{it}^T dollars of asset i with basis P_{it}, and (iii) short y_{it}^T dollars of bonds paying interest rate R_{ft}. At time $t+1$, the long share in part (i) equals $(1-\tau_{ct})\,c_{it}P_{it}$ of cash (the after-tax coupon payments) plus a long share with basis $\widehat{P}_i$, which has cash value $V_{it+1}(\widetilde{P}_{it+1}, \widehat{P}_i)$ according to the induction assumption. By Equation (22), the cash value of part (i) can be written as $P_{it}\widetilde{R}_{it}(\widehat{P}_i)$ (which equals $P_{it}R_{it}^H(\widehat{P}_i)$ and $P_{it}R_{it}^L(\widehat{P}_i)$ in the high- and low-return states, respectively). Each dollar of short position in part (ii) equals a cash value of $-(1-\tau_{ct})\,c_{it}$ (coupon payments owed) and shorting $(1/P_{it})$ shares of asset i with basis P_{it}, which has a cash value of $S_{it+1}(\widetilde{P}_{it+1}, P_{it})/P_{it}$ by the induction assumption. In addition, from Assumption 12, the investor needs to post one dollar as cash collateral earning only $\widehat{r}_{it}$, with a total loss of after-tax interest payments of $(R_{ft} - \widehat{R}_{it})$ dollars. By Equations (A3d) and (A3e), the return on the short position is $\widetilde{R}_{-it}(P_{it})$ (which equals $R^H_{-it}(P_{it})$ or $R^L_{-it}(P_{it})$). Thus, the net cash value of portfolio (i)–(iii) at time $t+1$ is

$$\widetilde{F}_i(\widehat{P}_i) = P_{it}\widetilde{R}_{it}(\widehat{P}_i) + x^T_{it}\widetilde{R}_{-it}(P_{it}) - y^T_{it}R_{ft}. \tag{A4}$$

Similarly, to realize the cash value of a short position in asset i with cost basis $\widehat{P}_i$, the investor can borrow $-S_{it}(P_{it}, \widehat{P}_i)$ dollars of cash to purchase x^T_{-it} dollars of the asset and invest the rest $(-S_{it}(P_{it}, \widehat{P}_i) - x^T_{-it} = -y^T_{-it})$ in bonds. In addition to his cash position $-S_{it}$, the investor is exposed to a portfolio that is (i′) short one share of asset i with basis $\widehat{P}_i$, (ii′) long x^T_{-it} dollars of asset i with basis P_{it}, and (iii′) long $-y^T_{-it}$ dollars of bonds with interest rate r_{ft}. Similar to Equation (A4), the net cash value of portfolio (i′)–(iii′) at time $t+1$ is

$$\widetilde{F}_{-i}(\widehat{P}_i) = P_{it}\widetilde{R}_{-it}(\widehat{P}_i) + x^T_{-it}\widetilde{R}_{it}(P_{it}) - y^T_{-it}R_{ft}. \tag{A5}$$

To prove the induction step, we need to prove that both portfolios (i)–(iii) and (i′)–(iii′) have a value of zero. We start with portfolio (i)–(iii) and show that the payoff of Equation (A4), $\widetilde{F}_i = F^H_i$, and F^L_i can be replicated using asset i and the bond. Specifically, if $F^H_i \geq F^L_i$, then $\widetilde{F}_i$ is equivalent to the payoff of (x_F, y_F) dollars of stocks (long position) and bonds, where

$$\begin{cases} x_F R^H_{it}(P_{it}) + y_F R_{ft} = F^H_i(\widehat{P}_i) \\ x_F R^L_{it}(P_{it}) + y_F R_{ft} = F^L_i(\widehat{P}_i) \end{cases} \Rightarrow \begin{cases} x_F(\widehat{P}_i) = \dfrac{F^H_i(\widehat{P}_i) - F^L_i(\widehat{P}_i)}{R^H_{it}(P_{it}) - R^L_{it}(P_{it})} > 0 \\ y_F(\widehat{P}_i) = \dfrac{F^L_i(\widehat{P}_i)R^H_{it}(P_{it}) - F^H_i(\widehat{P}_i)R^L_{it}(P_{it})}{R_{ft}\left(R^H_{it}(P_{it}) - R^L_{it}(P_{it})\right)}. \end{cases} \tag{A6}$$

The value of portfolio (i)–(iii) is $x_F + y_F$. If instead, $F^H_i < F^L_i$, then $\widetilde{F}_i$ is equivalent to the payoffs of (x_{-F}, y_{-F}) dollars of stocks (short position) and bonds, where

$$\begin{cases} x_{-F} R^H_{-it}(P_{it}) + y_{-F} R_{ft} = F^H_i(\widehat{P}_i) \\ x_{-F} R^L_{-it}(P_{it}) + y_{-F} R_{ft} = F^L_i(\widehat{P}_i) \end{cases} \Rightarrow \begin{cases} x_{-F}(\widehat{P}_i) = \dfrac{F^H_i(\widehat{P}_i) - F^L_i(\widehat{P}_i)}{R^H_{-it}(P_{it}) - R^L_{-it}(P_{it})} > 0 \\ y_{-F}(\widehat{P}_i) = \dfrac{F^L_i(\widehat{P}_i)R^H_{-it}(P_{it}) - F^H_i(\widehat{P}_i)R^L_{-it}(P_{it})}{R_{ft}\left(R^H_{-it}(P_{it}) - R^L_{-it}(P_{it})\right)} \end{cases} \tag{A7}$$

and the value of portfolio (i)–(iii) is $-x_{-F} + y_{-F}$.

Before we can calculate the value of portfolio (i)–(iii) (either $x_F + y_F$ or $-x_{-F} + y_{-F}$), we need to define the arbitrage-free rebate rate $\widehat{R}_{it}$. Consider the special case of $\widehat{P}_i = P_{it}$, then $V_{it}(P_{it}, P_{it}) = P_{it}$ holds trivially, and x^T_{it} and y^T_{it} in Equation (21) equal 1 and 0, respectively. If $F^H_i(P_{it}) \geq F^L_i(P_{it})$, then substituting Equation (A4) (with $x^T_{it}=1$ and $y^T_{it}=0$) into Equation (A6) and setting the replication cost $x_F(P_{it}) + y_F(P_{it})$ to zero yields the arbitrage-free rebate rate $\widehat{R}_{it}$. Simple algebra shows that the $\widehat{R}_{it}$ in Equation (A3f) solves the problem. If instead $F^H_i(P_{it}) < F^L_i(P_{it})$, combining Equations (A4) and (A7) while setting $-x_{-F}(P_{it}) + y_{-F}(P_{it})$ to zero yields the same result.

Given the rebate rate $\widehat{R}_{it}$, we can verify that the value of $x_F + y_F$ and $-x_{-F} + y_{-F}$ in Equations (A6) and (A7) is always zero once we plug in the definition of Equation (A4) and the portfolio (x^T_{it}, y^T_{it}) in Equation (21). Thus, the current value of portfolio (i)–(iii) for any $\widehat{P}_i$ is always zero, proving that V_{it} is the equivalent cash value for a long position in asset i with cost basis $\widehat{P}_i$.

Finally, we can construct replication portfolios similar to those in Equations (A6) and (A7) for payoffs in Equation (A5). Under the above rebate rate $\widehat{R}_{it}$, we can verify that the replication cost is also always zero, and so is the current value of portfolio (i′)–(iii′) for x^T_{-it} and y^T_{-it} defined in Equation (A3). Thus, a short position in asset i with cost basis $\widehat{P}_i$ is equivalent to S_{it} dollars of cash, proving the induction assumption.

Proof of Proposition 4

The proof is similar to those of Lemma 2 and Propositions 1 and 2, with the new replication portfolios defined in Lemma 6. The main difference is that holdings H_{t-1} and bases $\widehat{P}_{it}$ replace W_t^T as the state variable. Using backward induction, we can still show that the tax-deferred wealth is equivalent to a taxable wealth of $Z_t W_t^D$, which increases the taxable holding of risk-free assets and yields the effective taxable holding of $\widehat{H}_{t-1}$. Also, investors subtract from their taxable portfolio the taxable replication portfolio (x_{it}, y_{it}) for each dollar of asset i in the taxable account. We need to make sure that all newly created positions have a tax basis of P_{it} and that all existing positions retain their original tax bases. Note that the taxable strategy in Lemma 4 of postponing gains and realizing losses for all existing taxable positions is implicitly included in the optimal single-account strategy H_t^*, and is not explicitly reported in the proposition.

Proof of Proposition 5

We skip the proof for result (iii), since it is similar to that of Propositions 1 and 2. We prove results (i) and (ii) using backward induction. On the terminal date, $Z_t^M = 0$, since there is no more contribution and $Z_T = 1$. Both the consumption and portfolio results hold trivially. We assume that all results hold at $t+1$ and prove for time t.

Each dollar in the tax-deferred account is equivalent to Z_t dollars in the taxable account, defined in Equation (8). We show that investors always contribute the maximum allowed to their tax-deferred accounts, or $M_t = \overline{M}_t$. If not, they can increase their contribution by ϵ, following the optimal tax-deferred strategy in (ii) and shorting the replicating portfolio in the taxable account. They can match the risk of the original strategy and increase their taxable wealth by $\epsilon(Z_t - 1) \geq 0$. Hence, they always contribute the maximum allowed.

The present value of tax benefits for all future contributions Z_t^M has two components. First, by the induction assumption, contributions from $t+1$ to T are worth Z_{t+1}^M at time $t+1$, with a present value of Z_{t+1}^M / R_{ft+1} at t. Second, the investor is allowed to contribute $\overline{M}_t$ at t, yielding a tax benefit of $\overline{M}_t(Z_t - 1)$. Hence, Z_t^M satisfies Equation (28). Combining Z_t^M and the $\widehat{W}_t^T$ in Proposition 2 yields the new $\widehat{W}_t^T$ in Equation (27). Using similar arguments to those in the proof of Proposition 2, we can map the two-account problem to a taxable-account-only problem. The optimal consumption depends only on the new $\widehat{W}_t^T$, since investors can always borrow to finance their consumption.

Proof of Proposition 6

Conditioning on Θ_t^D and $\widetilde{Z}_{t+1}$, we can denote Θ_t^D in Equation (7) as Θ_t^{D*} and show that

$$J_t\big(W_t^T, W_t^D \mid \Theta_t^{D*}\big) = \mathrm{E}_{Z_{t+1}}\big[J_t\big(W_t^T, W_t^D \mid \Theta_t^{D*}, Z_{t+1}\big)\big], \tag{A8a}$$

$$= \mathrm{E}_{Z_{t+1}}\big[J_t\big(W_t^T + \big(\Theta_t^{D*\prime} z_t\big) Z_{t+1} W_t^D,\ 0\big)\big], \tag{A8b}$$

$$\geq \mathrm{E}_{Z_{t+1}}\big[J_t\big(W_t^T + \big(\Theta_t^{D\prime} z_t\big) Z_{t+1} W_t^D,\ 0\big)\big], \tag{A8c}$$

$$= \mathrm{E}_{Z_{t+1}}\big[J_t\big(W_t^T, W_t^D \mid \Theta_t^D, Z_{t+1}\big)\big] = J_t\big(W_t^T, W_t^D \mid \Theta_t^D\big), \tag{A8d}$$

where Equation (A8a) is the law of iterated expectations. Given that τ_{it} is constant during this period, z_t is well defined (as in Lemma 2). We can repeat the proof of Lemma 2 and take expectation over Z_{t+1} to derive Equation (A8b). From Proposition 1, Θ_t^{D*} maximizes the conditional value function inside the expectation of Equation (A8b) for any given Z_{t+1}. Taking expectation over Z_{t+1} yields Equation (A8c). Equation (A8d) follows similar arguments as Equations (A8a) and (A8b). Therefore, Θ_t^{D*} is the optimal tax-deferred portfolio in the current setting.

Given that Θ_t^{D*} maximizes J_t, we have

$$J_t\big(W_t^T, W_t^D\big) = J_t\big(W_t^T, W_t^D \mid \Theta_t^{D*}\big) = \mathrm{E}_{Z_{t+1}}\big[J_t\big(W_t^T + \hat{z}_t Z_{t+1} W_t^D,\ 0\big)\big], \tag{A9}$$

where the second equality is due to Equation (A8b) and $\hat{z}_t = \Theta_t^{D*\prime} z_t$ from Equation (8).

Supplementary data

Supplementary data are available online at http://rfs.oxfordjournals.org/.

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