

# Multiechelon Procurement and Distribution Policies for Traded Commodities

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We consider a firm that procures and distributes a commodity from spot and forward markets under randomly fluctuating prices; the commodity is distributed downstream to a set of nonhomogeneous retailers to satisfy random demand. We formulate a model that allows one to compute approximate, but near optimal, procurement and distribution policies for this system, and we explore the value of the commodity's market in providing managers with (a) additional flexibility in procurement and (b) information on price dynamics generated through the trading of futures contracts. Our results indicate that the presence of the commodity market and the information that it conveys may lead to significant reductions in inventory-related costs; however, to obtain these benefits, both the spot procurement flexibility and the term structure of prices generated by the commodity market must be incorporated in the formulation of the operating policy. Managerial insights on the procurement strategy as a function of variability in prices and demand are also discussed.

*Key words:* commodity procurement; commodity markets; marginal convenience yield; multiechelon inventory management

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## 1. Introduction

We consider a firm that procures a commodity from spot and forward markets at a central location and distributes it downstream to a set of non-identical retailers. Procurement and distribution of commodities differs from conventional goods distribution in two aspects: (1) spot (market) procurement is often available enhancing procurement flexibility, and (2) the trading of futures contracts provides important price information not available for conventional goods. Traded commodity prices evolve randomly, and the spreads between spot and futures prices determine the economic cost of holding the physical commodity. In our model this holding cost is random, and it is related to the concept of marginal convenience yield in the economics literature.<sup>1</sup>

In this paper, we develop procurement and distribution policies that incorporate the spot and futures price information observed in the commodity markets, and we show how this price information can be used to enhance the efficiency of a distributive supply chain. In particular, we focus on assessing the value of

spot procurement flexibility and the value of dynamically incorporating futures price information in procurement and distribution decisions in a two-echelon distribution system consisting of a single depot centralizing procurement and distributing the commodity to multiple retailers.

There is a vast operations management literature dealing with the class of single-depot multi-retailer distribution systems, and it provides good solutions for cases when demand is stationary, retailers are identical, and prices are constant. Nevertheless, McGavin et al. (1997) and others point out that inventory balancing assumptions commonly used in research models are not optimal for nonstationary demand and nonidentical retailers. On the other hand, there is a dearth of research explaining how the availability of a spot market affects the optimal procurement and distribution strategies of the above distributive systems; moreover, to the best of our knowledge, there is no published research on how to incorporate the price information generated in the commodity markets to design procurement and distribution policies for these systems.

Our research contribution is to address the gaps in the operations management literature stated above as

<sup>1</sup> This cost is also referred in the economics literature as the cost of carry (Working 1949).

follows: (1) We develop an optimization model and characterize the procurement and distribution policies for the case of nonidentical retailers in the context of one-warehouse and multiretailer paradigm; we derive these policies without making the inventory balancing assumption. (2) We quantify the value of the flexibility provided by the spot market in the procurement process and the value of incorporating the market-generated price information in decision making. One of our key methodological contributions is to use the marginal convenience yield information estimated from the prices observed at the commodity markets to compute the value of the Lagrange multiplier of the *allocation constraint* in order to obtain inventory procurement and distribution decisions *without* making a balancing assumption.

Our research findings indicate that there are important benefits derived from the possibility of procuring from the spot market. Moreover, we also identify cases in which it is critically important to also incorporate the stochastic nature of futures price information in the decision-making process. Our managerial insights potentially contribute to current practice by enhancing our understanding of the role and relevance of futures price information and spot markets in the management of commodities procurement and distribution decision making.

This research was motivated by the distribution of gasoline in a large, vertically integrated, oil company. The supply chain of gasoline in the United States is a very complex network that distributes around nine million barrels per day across over 160,000 retail gas stations spread across the country (Energy Information Administration (EIA) 2008). Gasoline is shipped from refineries to distribution terminals through a network of pipelines; it is then loaded on tanker trucks and shipped to the retailer network. In this research, we focus on the procurement of gasoline at the terminal and its direct distribution to the retailers. Advances in information technology have enabled the successful implementation of vendor management inventory (VMI) practices at the retailers (Worthen 2002); therefore, from a cost management perspective, this system can be treated as a vertically integrated channel.

Gasoline at the terminal can be procured using both forward and spot transactions. In the wake of supply disruptions or spikes in consumer demand, inventory at the terminal may fall below desirable levels, forcing terminal managers to procure gasoline through spot transactions; according to *CIO* magazine (Worthen 2002), up to 30% of the gasoline supply at Chevron is obtained through spot purchases. Hence, in this operating environment, it is pertinent for managers to understand the trade-offs involved in spot and forward procurement. Our model provides

insights to these trade-offs, and it assesses the value of spot procurement flexibility and the value of dynamically incorporating futures price information in procurement and distribution decision making. Because the modeling assumptions used in this paper adequately describe the procurement and distribution of other commodities, such as wheat, sugar, and industrial metals, our model and results also have potential relevance in these settings.

The literature on commodity markets and spot procurement as well as relevant literature on multiechelon inventory systems are discussed in §2. In §3 we describe in detail the optimization model, and we state the general properties of the optimal solution. These properties are utilized in §4 to devise, in conjunction with market-generated price information, an approximation to the optimal solution that is feasible to implement in large-scale systems. In §5 we conduct numerical experiments to generate managerial insights. Concluding remarks are presented in §6.

## 2. Related Literature

In this section, we briefly discuss literature on three main areas that this research touches upon. In §2.1 we introduce the economics literature related to the concept of marginal convenience yield. This is important because, as our analysis will show, the market-generated marginal convenience yield will impact optimal operational policies. In §2.2 we discuss the literature addressing the impact of commodity markets on supply chains. Finally, as our research problem falls under the general class of one-warehouse multiretailer problems, we briefly discuss this literature in §2.3.

### 2.1. Marginal Convenience Yield

To rationalize why commodity handlers and processors hold inventories when it is costly for them to do so, Kaldor (1939) reckoned that there must be a benefit associated with holding the physical commodity and he called this benefit a convenience yield. Later, Working (1948) further explained that commodity handlers and processors are willing to pay a positive holding cost "... for the sake of assurance against having their merchandising or manufacturing activities handicapped" (Working 1948, p. 28).

The economic cost of holding a traded commodity in inventory (i.e., the difference between its spot price minus its discounted futures price plus the cost of storage) is determined by the commodity markets, and it is called "marginal convenience yield" (see Pindyck 2004); to prevent arbitrage through storage, this marginal convenience yield must be nonnegative because otherwise economic agents can simultaneously buy the commodity on the spot, pay for

storage, and sell it forward while making a risk-free profit.

Our contribution to the emerging literature in the operations/finance interface is to show how information conveyed by financial markets for commodities can be used to support operational choices within a firm. In this paper we use the market-generated marginal convenience yield to approximately solve a stochastic dynamic program that determines a procurement and distribution policy for a traded commodity.

There is also a related body of literature in finance focusing on modeling the dynamics of commodity prices; this includes Schwartz (1997) and Schwartz and Smith (2000), which model commodity prices as a mean-reverting process with stochastic convenience yield. These type of price models are needed in the dynamic optimization of commodity procurement to describe the stochastic behavior of the forward curves over time. In §5 we use the model in Schwartz and Smith (2000) in our numerical experiments, and at the end of §5.3 we discuss in detail the special considerations required for using these stochastic models of commodity prices in operations management models dealing with physical procurement.

## 2.2. Commodity Markets and Procurement

In addition to the operations management research on commodities procurement, we can identify two additional research streams that precede it and share some of its relevant features: (1) models with multiple modes of supply, in which faster procurement alternatives are available at a premium cost, and (2) inventory models, in which prices are subject to stochastic fluctuations. The models in Fukuda (1964), Whittemore and Saunders (1977), Moinezadeh and Lee (1989), Chiang and Gutierrez (1998), Lawson and Porteus (2000), Tagaras and Vlachos (2001), and Huggins and Olsen (2003), among others, study systems where a faster procurement alternative is available at a premium under a deterministic price framework. Studies of procurement policies under fluctuating procurement prices include Golabi (1985), Gavirneni (2004), and Gavirneni and Morton (1999).

The procurement of traded commodities has recently received considerable research attention; this literature stream includes Akella et al. (2002), Yi and Scheller-Wolf (2003), Seifert et al. (2004), Martínez-de-Albéniz and Simchi-Levi (2006), Fu et al. (2010), Dong and Liu (2007), Goel and Gutierrez (2011), Secomandi (2010a), and Devalkar et al. (2011). Studies related to oil and natural gas include Tayur and Yang (2002), Secomandi (2010b), Wu and Chen (2010), and Lai et al. (2011). For a detailed literature review on impact of spot markets on supply chains, refer to Haksöz and Seshadri (2007).

The papers discussed above study commodity procurement with a focus on a single location. Our paper, on the other hand, addresses the role of spot procurement in the context of a multiechelon two-level distribution system consisting of a single-depot supplying multiple retailers. In a related paper, Reiman et al. (1999) study the distribution of gasoline with a focus on transportation policies within a queuing model framework.

## 2.3. Multiechelon Inventory Systems

The complexity of obtaining optimal solutions for multiechelon systems involving a central procurement facility and multiple downstream locations is well understood (Clark and Scarf 1960). To deal with this problem, researchers have repeatedly resorted to approximations. In this regard, a “balancing of inventory” assumption is very common; this entails assuming that through feasible shipments from the warehouse to the retailers it is possible to bring all the retailers to a common normalized inventory level. For detailed discussions on this topic, we refer readers to Eppen and Schrage (1981), Federgruen and Zipkin (1984a), Jonsson and Silver (1987), Jackson (1988), Erkip et al. (1990), and references therein. Zipkin (1984) addresses the issue of imbalances in inventory and postulates that for a system in which all retailers have equal coefficients of variation of demand, a myopic allocation of stock among retailers yields a near optimal solution. The nonoptimality of balancing policies for the case of nonidentical retailers is discussed in detail by Federgruen and Zipkin (1984b), Jackson and Muckstadt (1989), and McGavin et al. (1997). Our model contributes to this research stream by obtaining procurement and distribution policies for multiechelon distribution system without making a balancing assumption.

## 3. Multiretailer Distribution Model

In §3.1 we define in detail the procurement and distribution model, and in §3.2 we characterize procurement policies from spot and forward markets.

### 3.1. Mathematical Model

The procurement and distribution model consists of a central warehouse and  $N$  downstream retail locations. This model is cast within the framework of a periodic review inventory system; the overall objective is to devise operating policies that minimize the present value of expected costs over a planning horizon of  $T$  time periods. The decision variables, at each time period  $t$ , are the spot and forward procurement decisions at the upper echelon and the deployment of inventory at each of the different retail locations. The information available to make these decisions includes the commodity's price evolution

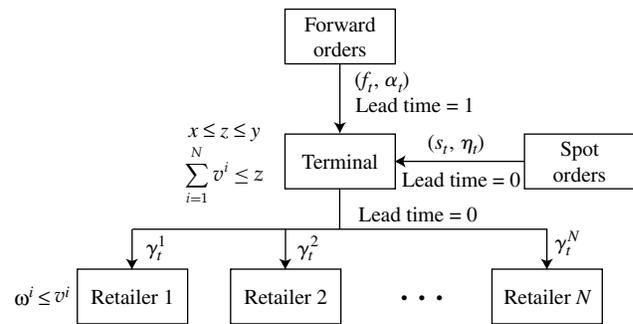
(spot and futures market prices) up to time  $t$ , the demand distribution at each retailer, the initial inventory at each retail location, and the initial inventory at upper echelon.

There are two modes of procurement, forward contracts and spot purchases. Commodity bought through a forward contract at review period  $t$  arrives at the beginning of  $t + 1$  while spot purchases arrive in the same review period. Moreover, we assume that spot and forward procurement decisions are made before the period's demand is observed. We assume the transportation cost associated with forward procurement, denoted as  $\alpha_t$ , is smaller than the spot procurement transportation cost, denoted as  $\eta_t$ , because in spot procurement the commodity must be transported with a shorter notice. After the execution of spot and forward decisions, inventory is shipped downstream to the retailers before the demand for this period is realized, and it is received by the retailer in the same review period. We denote the unit cost of shipping the commodity from the terminal to the location of retailer  $i$  as  $\gamma_i^i$ .

The initial echelon inventory at the upper echelon is denoted as  $x$  (it includes inventory at the terminal and at the retailers). This echelon inventory is raised to  $z \geq x$  through the procurement of  $(z - x)$  units from the spot market; then the echelon inventory position is raised to  $y \geq z$  by a forward procurement quantity  $(y - z)$ . There are  $N$  nonhomogeneous retailers, which possibly differ in their demand probability distribution, their storage costs, and their shortage penalty costs. The initial inventory of retailer  $i$  is denoted as  $w^i$ , and after distribution decisions are made it is raised to  $v^i \geq w^i$ ; additionally, the total inventory allocated to the retailers must not exceed the upper-echelon inventory,  $\sum_{i=1}^N v^i \leq z$ .

The spot price of the commodity at time  $t$  is denoted as  $s_t$ , and it is assumed to evolve stochastically as a function of *only* the information contained in a state vector  $\mathcal{S}_t$ . Thus the current price  $s_t$ , as well as the transition probabilities of the commodity price process from  $s_t$  to its value at  $t + \Delta t$  denoted as  $s_{t+\Delta t}$ , can be obtained from the information contained in  $\mathcal{S}_t$ ; to simplify our exposition, we assume  $\Delta t = 1$ . We assume the firm is a price taker in the commodity market; hence, the state  $\mathcal{S}_t$  consists of information related to price history, and it excludes any information specific to the firm. We assume the forward procurement contracts are priced at the futures price observed in the commodity's market.<sup>2</sup> The commodity is later shipped to the lower-echelon retailers, as shown in Figure 1.

Figure 1 Two-Echelon Procurement and Distribution System



To preclude risk-free arbitrage opportunities, we use the risk-neutral valuation approach (Duffie 2001) and define the futures price for time  $\tau > t$ , observed at time  $t$  as the expected spot price at time  $\tau$  under the risk-neutral probability measure,  $f_{t,\tau} = E_{\mathcal{S}_t}^{\mathbb{Q}} [s_{\tau}]$ ; the superscript  $\mathbb{Q}$  is used to denote that the expectation is taken using the risk-neutral probability measure.<sup>3</sup> Consistent with the real options literature, we assume the market trades enough financial instruments to replicate the uncertain price factors so that we can assume a unique risk-neutral probability measure for the evolution of  $\mathcal{S}_t$  can be found. In the interest of generality, at this point we abstract from any specific models of prices, and hence the functional forms for either the historical or the risk-neutral probability measures will remain unspecified because our analysis and results do not require it. At this point we just need to assume that both probability measures exist and that the expectation  $E_{\mathcal{S}_{t+1}}^{\mathbb{Q}} [g]$  exists and is finite for any finite real valued function  $g$ . In §5.1 we provide an example of a stochastic model of commodity prices in which  $\mathcal{S}_t$  consists of two factors.

We assume warehousing and penalty costs are paid at the end of each period; however, for notational simplicity we assume the corresponding unit costs are expressed as the present value at the beginning of review period  $t$ . The cost of each unit of unsatisfied demand for retailer  $i$  is denoted as  $p_i^i$ , the unit cost of storage at retailer  $i$  is  $h_i^i$ , and  $h_i$  is the unit storage cost at the upper echelon that is assumed to be the same for all the firms in the industry. If the on-hand inventory is  $v$ , the resulting expected storage and penalty costs for retailer  $i$  at time  $t$ , denoted as  $L_t^i(v)$  as a function of  $v$ , are given by  $L_t^i(v) = \int_0^v h_i^i(v - \xi) \phi_i^i(\xi) d\xi + \int_v^\infty p_i^i(\xi - v) \phi_i^i(\xi) d\xi$ , where  $\phi_i^i(\xi)$  denotes the probability density function of demand,  $\xi$ , at retailer  $i$  at time period  $t$ , and  $\Phi_i^i(\xi)$  denotes the corresponding probability distribution function.

<sup>2</sup> Forward prices are equivalent to futures prices as far as interest rates are constant over the life of the contract (Hull 2003).

<sup>3</sup> We will denote the one-period-ahead futures prices as  $f_t = f_{t,t+1}$ , and we omit the subscript in the expectation and denote  $f_t = E^{\mathbb{Q}} [s_{t+1}]$  when no ambiguity will result from doing it.

Our model makes three critical assumptions about demand: First, we assume demand is independent of price at any time  $t$ . Empirical studies (see, for example, Graham and Glaister 2002 and references therein) have found that although demand for gasoline is elastic over the long term, it is relatively inelastic over the short term (defined as a year); hence, this assumption is appropriate for a procurement and inventory management model where the relevant planning horizon is often significantly shorter than a year. Second, we assume demands for the commodity in different periods are independent but do not require to assume demand independence across locations. Third, we further assume demand is uncorrelated with the returns of the market portfolio, and thus demand risk is fully diversifiable; this assumption is supported by the findings in Berling and Rosling (2005). Under these two assumptions, the risk-neutral distribution of demand is equal to the historical (true) probability distribution of demand, and the joint risk-neutral probability distribution of the price process and demand is obtained as the product of the marginals; this enables us to factor demand and price-related expectations.

In addition, demands of different periods are independent random variables, and unsatisfied demand is backlogged; we make these two assumptions for tractability. The vector of demand realizations at each retailer in period  $t$  is denoted as  $\vec{\xi}_t = (\xi_t^i)$ , and we denote the aggregate demand of the network as  $\zeta_t = \sum_{i=1}^N \xi_t^i$ . Let  $\phi_t(\zeta_t)$  denote the probability density function of the aggregate demand  $\zeta_t$  and  $\Phi_t(\zeta_t)$  the corresponding distribution function. The vector of inventory levels at the retailers is represented by  $\mathbf{w} = (w^i)$ , and the allocation of inventory to respective retailers is denoted by  $\mathbf{v} = (v^i)$  for retailers  $i = 1, \dots, N$ . The present value of the expected cost at time  $t$  is represented by the stochastic dynamic program with state variables,  $\mathcal{S}_t, x$ , and  $\mathbf{w}$ , defined by the following problem statement.

**Problem ( $V_t$ ):**

$$V_t(\mathcal{S}_t, x, \mathbf{w}) = \min_{z, y, v^i} \left\{ (s_t + \eta_t)(z - x) + \beta(f_t + \alpha_t)(y - z) + \sum_{i=1}^N \gamma_t^i(v^i - w^i) + \sum_{i=1}^N L_t^i(v^i) + h_t \left( z - \sum_{i=1}^N v^i \right) + \beta E_{\vec{\xi}_t}^{\mathbb{Q}} E_{\mathcal{S}_t}^{\mathbb{Q}} [V_{t+1}(\mathcal{S}, y - \zeta_t, \mathbf{v} - \vec{\xi}_t)] \right\} \quad (1)$$

subject to  $x \leq z \leq y$

$$\sum_{i=1}^N v^i \leq z$$

$$w^i \leq v^i \quad \text{for } i = 1, 2, \dots, N,$$

and  $V_{T+1}(\mathcal{S}_{T+1}, x, \mathbf{w}) = (s_{T+1} + \eta_{T+1})(-x)^+ - (s_{T+1} - \eta_{T+1})(x)^+$ , where  $(x)^+ = \max(x, 0)$ . The present value of cash flows generated next period is obtained using a risk-free multiplicative discount factor  $\beta = e^{-r\Delta t}$ , where  $r$  is the risk-free discount rate.

The first term in the cost function (1) is the cost of procurement from the spot market, the second term represents the cost of procurement through a forward contract, the third term is the aggregate cost of transportation of the commodity from the terminal to the respective retailers, the fourth term is the loss function of the retailers, the fifth term is the holding cost of stock at the upper echelon, and the sixth term is the discounted cost to go in the future period. To simplify the analysis of the model, below we introduce the operator  $\mathcal{H}_t$ . Let  $g$  be a real valued function of  $(\mathcal{S}, x, \mathbf{w})$ ; then  $\mathcal{H}_t[g]$  is defined as follows:

$$\begin{aligned} \mathcal{H}_t[g](\mathcal{S}_t, x, y, z, \mathbf{w}, \mathbf{v}) &= (s_t + \eta_t)(z - x) + \beta(f_t + \alpha_t)(y - z) \\ &+ h_t \left( z - \sum_{i=1}^N v^i \right) + \sum_{i=1}^N \gamma_t^i(v^i - w^i) + \sum_{i=1}^N L_t^i(v^i) \\ &+ \beta E_{\vec{\xi}_t}^{\mathbb{Q}} E_{\mathcal{S}_t}^{\mathbb{Q}} [g(\mathcal{S}, y - \zeta_t, \mathbf{v} - \vec{\xi}_t)]. \end{aligned} \quad (2)$$

For notational simplicity we will write the operator  $\mathcal{H}_t[g](\mathcal{S}_t, x, y, z, \mathbf{w}, \mathbf{v})$  as  $\mathcal{H}_t[g]$ . Using the operator  $\mathcal{H}_t$  we can write the program defined in (1) as

$$\begin{aligned} V_t(\mathcal{S}_t, x, \mathbf{w}) &= \min \mathcal{H}_t[V_{t+1}] \\ \text{s.t. } &x \leq z \leq y \\ &\sum_{i=1}^N v^i \leq z \\ &w^i \leq v^i \quad \text{for } i = 1, \dots, N. \end{aligned} \quad (3)$$

Before proceeding to characterize the optimal procurement and distribution policy in §3.2, we will formalize the functional properties of  $\mathcal{H}_t$  and  $V_t$  in the three lemmas below. Proofs to lemmas and theorems can be found in Appendix A whenever they are not included in the body of the paper.

**LEMMA 1 (ISOTONICITY OF  $\mathcal{H}_t$ ).** *If  $g_1$  and  $g_2$  are real valued functions of the appropriate dimension<sup>4</sup> such that  $g_1 \leq g_2$  pointwise, then  $\mathcal{H}_t[g_1] \leq \mathcal{H}_t[g_2]$ .*

**LEMMA 2 (PRESERVATION OF CONVEXITY).** *If  $g(\mathcal{S}, x, \mathbf{w})$  is convex in  $(x, \mathbf{w})$ , then  $\mathcal{H}_t[g]$  is convex in  $(x, y, z, \mathbf{v}, \mathbf{w})$ .*

**LEMMA 3 (CONVEXITY OF  $V_t$ ).** *The function  $V_t$  is convex in  $(x, \mathbf{w})$  for every  $\mathcal{S}_t$ .*

<sup>4</sup>By this we mean  $g_k: \mathbb{R}^{N+1+|\mathcal{S}|} \rightarrow \mathbb{R}$ , for  $k = 1, 2$  with  $|\mathcal{S}|$  denoting the number of elements in vector  $\mathcal{S}$ .

### 3.2. Characterization of Optimal Procurement Policies

The analysis in this subsection develops insights on procurement policies from spot and forward markets. In this regard, we obtain the following set of first-order conditions for Problem  $(V_t)$ :

$$s_t + h_t - \beta f_t + \eta_t - \beta \alpha_t - \lambda_t - \mu_t + \pi_t = 0, \quad (4)$$

$$\beta f_t + \beta \alpha_t + \beta \frac{\partial}{\partial y} [E_{\xi_t} E_{\mathcal{S}|\mathcal{S}_t}^Q [V_{t+1}(\mathcal{S}, y - \zeta_t, \mathbf{v} - \vec{\xi}_t)]] - \pi_t = 0, \quad (5)$$

$$\gamma^i - h_t + \frac{\partial}{\partial v^i} [L_t(v^i)] + \beta \frac{\partial}{\partial v^i} [E_{\xi_t} E_{\mathcal{S}|\mathcal{S}_t}^Q [V_{t+1}(\mathcal{S}, y - \zeta_t, \mathbf{v} - \vec{\xi}_t)]]$$

$$+ \lambda_t - \theta_t^i = 0 \quad \text{for } i = 1, \dots, N, \quad (6)$$

$$\mu_t(z - x) = 0, \quad (7)$$

$$\lambda_t \left( z - \sum_{i=1}^N v^i \right) = 0, \quad (8)$$

$$\pi_t(y - z) = 0, \quad (9)$$

$$\theta_t^i(v^i - w^i) = 0 \quad \text{for } i = 1, \dots, N. \quad (10)$$

Equations (4), (5), and (6) represent the first-order conditions for decision variables,  $z$ ,  $y$ , and  $v^i$ , respectively. Equations (7)–(10) are the complementary slackness conditions, and  $\mu_t$ ,  $\lambda_t$ ,  $\pi_t$ , and  $\theta_t^i$  are the period- $t$  nonnegative Lagrange multipliers associated with constraints  $z \geq x$ ,  $z \geq \sum_{i=1}^N v^i$ ,  $y \geq z$ , and  $v^i \geq w^i$ , respectively. Before proceeding, it will be useful to define  $\delta_t$  as follows:

$$\delta_t = s_t + h_t - \beta f_t + \eta_t - \beta \alpha_t. \quad (11)$$

Assuming the price process allows no arbitrage opportunities implies  $s_t + h_t - \beta f_t \geq 0$  because otherwise economic agents can simultaneously buy on the spot market at  $s_t$ , sell forward at a price of  $f_t$ , pay  $h_t$  for storage between  $t$  and  $t + 1$ , and earn a risk-free profit in the transaction.<sup>5</sup> Our assumptions regarding the relative magnitude of procurement logistics costs are  $\eta_t - \beta \alpha_t > 0$ ; hence, we have  $\delta_t > 0$ . At each time  $t$  the observed value of  $\delta_t$  will be useful in estimating the value of the Lagrange multiplier associated with the allocation constraint  $z > \sum_{i=1}^N v^i$ ,  $\lambda_t$ , and it plays a pivotal role in making upper-echelon inventory allocation decisions to the retailers in the lower echelon.

Let  $z^*$ ,  $y^*$ , and  $v^{i*}$  denote the optimal solution (at time  $t$ ) to Problem  $(V_t)$ , where  $z^*$  and  $y^*$  denote

the optimal echelon inventory after procurement from spot and forward market, respectively, and  $v^{i*}$  denotes the optimal inventory at the retail locations. As a consequence of conditions (4)–(10) we can obtain the properties formalized in the following theorem.

**THEOREM 1 (STRUCTURAL PROPERTIES OF THE LAGRANGE MULTIPLIERS).** *At any period  $t$ , the optimal procurement policy at the upper echelon  $(y^*, z^*)$  and the values of Lagrange multipliers  $(\lambda_t, \mu_t, \pi_t)$  are related as follows:*

- (a) If  $y^* > z^* > x$ , then  $\pi_t = 0$ ,  $\mu_t = 0$ , and  $\lambda_t = \delta_t$ .
- (b) If  $y^* > z^* = x$ , then  $\pi_t = 0$ ,  $\mu_t \geq 0$ , and  $\lambda_t \leq \delta_t$ .
- (c) If  $y^* = z^* > x$ , then  $\pi_t \geq 0$ ,  $\mu_t = 0$ , and  $\lambda_t > \delta_t$ .
- (d) If  $y^* = z^* = x$ , then  $\pi_t \geq 0$ ,  $\mu_t \geq 0$ , and  $\lambda_t \geq 0$ .

Theorem 1(a) implies that when it is optimal to procure from both spot and forward markets, the value of the Lagrange multiplier  $\lambda_t$ , associated with the allocation constraint (8), must be equal to the marginal convenience yield  $\delta_t$ . Theorem 1(b) states that if there is a positive procurement from the forward market, but it is not optimal to procure from the spot market, then the value of the allocation constraint's Lagrange multiplier is bounded from above by  $\delta_t$ . According to Theorem 1(c) an optimal policy that prescribes spot, but no forward procurement,  $y^* = z^* > x$ , implies  $\lambda_t > \delta_t$ ; this occurs if demand is anticipated to decrease in subsequent periods or if the marginal convenience yield for period  $t + 1$  is anticipated to increase,  $E_{\mathcal{S}_{t+1}|\mathcal{S}_t}^Q [\delta_{t+1}] > \delta_t$ . The properties highlighted in Theorem 1 will be used in §4.2 to develop an approximation to the optimal operating policy.

The following proposition formalizes an intuitive, but important, property of optimal procurement and distribution policies; this result is an immediate consequence of the lack of arbitrage opportunities in the commodity price process and of the model's value (present value of cost) optimization objective.

**PROPOSITION 1 (SPOT PROCUREMENT IS ONLY FOR DISTRIBUTION).** *In an optimal policy, if there is procurement from the spot market,  $z^* > x$ , then all inventory in the system must be allocated to the retail locations,  $z^* = \sum_{i=1}^N v^{i*}$ .*

An alternative way to state the above proposition is that under an optimal operating policy, we will see physical inventory held at the upper echelon only in time periods in which there is no procurement from the spot market.

## 4. Lagrangian Relaxation Approach

As the number of retailers increases, the *curse of dimensionality* makes it very difficult to obtain an optimal solution to Problem  $(V_t)$ , and we must rely on

<sup>5</sup> As in this argument the commodity may not have to be transported; the terms  $\eta_t$  and  $\alpha_t$  (which are firm-specific) are not part of the (market-wide) no-arbitrage argument.

approximations. In §4.1 we formulate and analyze a Lagrangian relaxation of Problem  $(V_t)$  and show it can be decomposed into a sum of  $N + 1$  simpler dynamic programs. In §4.2 we use the Lagrangian relaxation model to develop approximate solutions to Problem  $(V_t)$ .

#### 4.1. Model Formulation

Problem  $(V_t)$  belongs to the family of *weakly coupled* dynamic programs, where a complex multidimensional dynamic program can be reformulated as a number of lower-dimensional programs linked together by a common constraint. In Problem  $(V_t)$  the *allocation constraint*,  $z \geq \sum_{i=1}^N v^i$ , links the stocking decisions at each retailer to overall system-wide procurement. By dualizing this constraint, conditional on the value of the Lagrange multiplier, we will be able to decouple the program into  $N + 1$  simpler programs, one for each retailer, and an additional program for the upper echelon. Then we use this decoupled Lagrangian relaxation to obtain approximations on the cost function of the original problem. Other researchers have used the Lagrangian relaxation methodology to approximate weakly coupled dynamic programs; see, for example, Adelman and Mersereau (2008) and Caro and Gallien (2007). However, differing from the above literature, in which the value of the multiplier is determined as a function of internal system parameters, in this model the value of the multiplier is obtained using both internal system parameters as well as information obtained from the commodity's market.

If we were to dualize the allocation constraint by subtracting the term  $\lambda_t(z - \sum_{i=1}^N v^i)$  from the objective function in (1), in an optimal solution, the value of the Lagrange multiplier  $\lambda_t$  at each period  $t$  will be a function of the commodity price process,  $\mathcal{S}_t$ ; the inventory position at the upper echelon; and the inventory position at each of the  $N$  retailers. To simplify computations, in our relaxation we allow the Lagrange multiplier to become a problem parameter that can change from one period to the next, but it remains constant across all values of the state space in each period.

We denote as  $\mathbf{a} = (a_1, a_2, \dots, a_T)$  the vector of Lagrange multipliers where each component  $a_t$  corresponds to the scalar value of the Lagrange multiplier that we associate with the allocation constraint on the corresponding time period; hence, the parameter  $a_t$  is our approximation of the value of the Lagrange multiplier  $\lambda_t$  in an optimal solution,  $a_t \approx \lambda_t$ . Let us denote as  $\mathbf{a}_t$  the subvector of  $\mathbf{a}$  that includes the Lagrange multipliers from time  $t$  through  $T$ ,  $\mathbf{a}_t = (a_t, \dots, a_T)$ . Our analysis below assumes  $\mathbf{a}_t \geq 0$  and it is fixed. In the rest of this section we will establish fundamental properties of the relaxation; then in §4.2 we will

use these properties to approximate the optimal cost function  $V_t$ . Below we define formally the Lagrangian relaxation problem for a given value of the parameter vector  $\mathbf{a}$ .

**Problem  $(V^L)$ :**

$$\begin{aligned} V_t^L(\mathcal{S}_t, x, \mathbf{w}) &= \min \mathcal{H}_t^L[V_{t+1}^L] \\ \text{s.t. } &x \leq z \leq y \\ &w^i \leq v^i \quad \text{for } i = 1, \dots, N, \end{aligned} \quad (12)$$

where  $V_{T+1}^L(\mathcal{S}_t, x, \mathbf{w}) = (s_{T+1} + \eta_{T+1})(-x)^+ - (s_{T+1} - \eta_{T+1})(x)^+$  and the cost to go operator for the Lagrangian relaxation,  $\mathcal{H}_t^L$ , is defined as

$$\mathcal{H}_t^L[V_{t+1}^L] = \mathcal{H}_t[V_{t+1}^L] - a_t \left( z - \sum_{i=1}^N v^i \right). \quad (13)$$

It is clear from the above recursive definitions that both  $\mathcal{H}_t^L[V_{t+1}^L]$  and  $V_t^L$  are unaffected by  $(a_1, \dots, a_{t-1})$  (i.e., they depend only on the subvector  $\mathbf{a}_t$ ). It is immediately clear that  $\mathcal{H}_t^L[V]$  is also isotonic in  $V$  and convex in  $(x, y, z, \mathbf{v}, \mathbf{w})$  as established in Lemmas 1 and 2 for  $\mathcal{H}_t$ . We will refer to the optimal solution to Problem  $(V^L)$  as LR-optimal and denote it as  $\underline{z}^*$ ,  $\underline{y}^*$ , and  $\underline{v}^*$ .

**THEOREM 2 (LOWER BOUND ON  $V_t$ ).** *The Lagrangian relaxation  $V_t^L$  is a lower bound on  $V_t$  for any multiplier subvector  $\mathbf{a}_t \geq 0$ .*

We denote  $\max(x, y) = x \vee y$  and  $\min(x, y) = x \wedge y$ . The following result is a special case of Karush (1959), and it will be used repeatedly in the paper. We formalize it as a lemma for ease of reference.

**LEMMA 4.** *If a function  $g: \mathbb{R} \rightarrow \mathbb{R}$  is convex, then  $\min_{x \leq y} g(y) = g(x \vee y^*)$ , where  $y^*$  is the unconstrained minimizer of  $g$  and  $g(x \vee y^*)$  is nondecreasing and convex.*

The next theorem uses Lemma 4 to decompose the Lagrangian relaxation function  $V_t^L(\mathcal{S}_t, x, \mathbf{w})$  into a sum of functions of lower dimensionality. We use a semicolon to separate the state and decision variables from the subvector of  $\mathbf{a}$  parameterizing the problem; for example, the notation  $J_t^2(\mathcal{S}_t, y; \mathbf{a}_{t+1})$  indicates the function  $J_t^2$  is evaluated for the different values of  $(\mathcal{S}_t, y)$  using a specific value of the subvector  $\mathbf{a}_{t+1}$ . This distinction is important for our discussion below and subsequent algorithmic development.

**THEOREM 3 (DECOMPOSITION OF  $V_t^L$ ).** *The cost function  $V_t^L(\mathcal{S}_t, x, \mathbf{w})$  is convex in  $(x, \mathbf{w})$  for any  $\mathbf{a}_t \geq 0$  and can be decomposed as  $V_t^L(\mathcal{S}_t, x, \mathbf{w}) = H_t(\mathcal{S}_t, x; \mathbf{a}_t) + \sum_{i=1}^N G_t^i(w^i; \mathbf{a}_t)$ , where  $H_t$  is convex in  $x$  and  $G_t^i$  is convex in  $w^i$  for all  $i$  and for any  $\mathbf{a}_t \geq 0$ .*

**PROOF.** The validity of the theorem for time  $T + 1$  is immediate from (12) by setting  $H_{T+1}(\mathcal{S}_t, x; \cdot) = (s_{T+1} + \eta_{T+1})(-x)^+ - (s_{T+1} - \eta_{T+1})(x)^+$ , and  $G_{T+1}^i = 0$ ,

for all  $i$ . We proceed inductively by assuming that Theorem 3 holds for time  $t + 1$ ; thus,  $V_{t+1}^L(\mathcal{S}_t, x, \mathbf{w}) = H_{t+1}(\mathcal{S}_t, x; \mathbf{a}_{t+1}) + \sum_{i=1}^N G_{t+1}^i(w^i; \mathbf{a}_{t+1})$ .

Hence, it follows from (13) and (2) that  $\mathcal{H}_t^L[V_{t+1}^L]$  in (12) can be written as

$$\mathcal{H}_t^L[V_{t+1}^L] = J_t^1(\mathcal{S}_t, z; a_t) + J_t^2(\mathcal{S}_t, y; \mathbf{a}_{t+1}) + \sum_{i=1}^N F_t^i(v^i; \mathbf{a}_{t+1}) - (s_t + \eta_t)x - \sum_{i=1}^N \gamma_t^i w^i, \quad (14)$$

$$J_t^1(\mathcal{S}_t, z; a_t) = (s_t + h_t - \beta f_t + \eta_t - \beta \alpha_t - a_t)z = (\delta_t - a_t)z, \quad (15)$$

$$J_t^2(\mathcal{S}_t, y; \mathbf{a}_{t+1}) = \beta(f_t + \alpha_t)y + \beta E_{\xi_t} \mathbb{E}_{\mathcal{S}_t | \mathcal{S}_t}^Q [H_{t+1}(\mathcal{S}_t, y - \zeta_t; \mathbf{a}_{t+1})], \quad (16)$$

$$F_t^i(v^i; \mathbf{a}_{t+1}) = L_t^i(v^i) + (\gamma_t^i + a_t - h_t)v^i + \beta E_{\xi_t} G_{t+1}^i(v^i - \xi_t^i; \mathbf{a}_{t+1}). \quad (17)$$

It follows from the induction hypothesis that  $J_t^2(\mathcal{S}_t, y; \mathbf{a}_{t+1})$  is convex in  $y$  and the functions  $F_t^i$  are convex in  $v^i$  for all values of  $\mathbf{a}_{t+1}$ . Hence,  $\mathcal{H}_t^L[V_{t+1}^L]$  is convex in  $(x, y, z, \mathbf{v}, \mathbf{w})$ . Now we can write Equation (12) as

$$V_t^L(\mathcal{S}_t, x, \mathbf{w}) = \min_{x \leq z} \left\{ J_t^1(\mathcal{S}_t, z; a_t) + \min_{z \leq y} J_t^2(\mathcal{S}_t, y; \mathbf{a}_{t+1}) \right\} + \sum_{i=1}^N \min_{w^i \leq v^i} F_t^i(v^i; \mathbf{a}_{t+1}) - (s_t + \eta_t)x - \sum_{i=1}^N \gamma_t^i w^i. \quad (18)$$

It follows from Lemma 4 that  $\min_{z \leq y} J_t^2(\mathcal{S}_t, y; \mathbf{a}_{t+1}) = J_t^2(\mathcal{S}_t, y^u \vee z; \mathbf{a}_{t+1})$ , where  $y^u$  is the unconstrained minimizer of  $J_t^2(\mathcal{S}_t, y; \mathbf{a}_{t+1})$  and  $J_t^2(\mathcal{S}_t, y^u \vee z; \mathbf{a}_{t+1})$  is convex and nondecreasing. Applying Lemma 4 again, we can write

$$\begin{aligned} & \min_{x \leq z} \left\{ J_t^1(\mathcal{S}_t, z; a_t) + \min_{z \leq y} J_t^2(\mathcal{S}_t, y; \mathbf{a}_{t+1}) \right\} \\ &= \min_{x \leq z} \left\{ J_t^1(\mathcal{S}_t, z; a_t) + J_t^2(\mathcal{S}_t, y^u \vee z; \mathbf{a}_{t+1}) \right\} \\ &= J_t^1(\mathcal{S}_t, z^u \vee x; a_t) + J_t^2(\mathcal{S}_t, y^u \vee z^u \vee x; \mathbf{a}_{t+1}), \quad (19) \end{aligned}$$

where  $z^u$  is the unconstrained minimizer of the function  $(J_t^1(\mathcal{S}_t, z; a_t) + J_t^2(\mathcal{S}_t, y^u \vee z; \mathbf{a}_{t+1}))$ ; it follows from Lemma 4 that the function  $(J_t^1(\mathcal{S}_t, z^u \vee x; a_t) + J_t^2(\mathcal{S}_t, y^u \vee z^u \vee x; \mathbf{a}_{t+1}))$  is convex and nondecreasing in  $x$  for every  $\mathbf{a}_t$ . Now define

$$H_t(\mathcal{S}_t, x; \mathbf{a}_t) = J_t^1(\mathcal{S}_t, z^u \vee x; a_t) + J_t^2(\mathcal{S}_t, y^u \vee z^u \vee x; \mathbf{a}_{t+1}) - (s_t + \eta_t)x, \quad (20)$$

$$G_t^i(w^i; \mathbf{a}_t) = \min_{w^i \leq v^i} \{ F_t^i(v^i; \mathbf{a}_t) - \gamma_t^i w^i \}. \quad (21)$$

Hence,  $V_t^L(\mathcal{S}_t, x, \mathbf{w}) = H_t(\mathcal{S}_t, x; \mathbf{a}_t) + \sum_{i=1}^N G_t^i(w^i; \mathbf{a}_t)$ ; where the functions  $G_t^i$  are convex in  $w^i$ ,  $H_t$  is convex in  $x$ , and  $V_t^L$  is convex in  $(x, \mathbf{w})$  for any given  $\mathbf{a}_t \geq 0$ .  $\square$

For any given multiplier vector  $\mathbf{a}$ , Equations (21) and (17) define  $N$  independent single location stochastic inventory dynamic programs. Similarly, Equation (20) together with (15) and (16) define a stochastic dynamic program that determines LR-optimal spot and forward procurement decisions. Hence, the proof of Theorem 3 above elucidates an algorithm to decouple the larger dynamic program for the entire system into  $N + 1$  smaller dynamic programs, one for each retailer and one for the upper echelon. This result will be used extensively in our procedure to use the LR-optimal solution to approximate the optimal solution to Problem  $(V_t)$ . However, because a key part of the approximation consists of the judicious selection of the multiplier vector  $\mathbf{a}$ , before we introduce this approximation procedure, we need to establish the following analytical properties relating the LR-optimal solution to changes in the value of  $\mathbf{a}$ .

**LEMMA 5 ( $F_t^i$  AND  $G_t^i$  ARE SUPERMODULAR).** *Let  $a_\tau$  be the scalar value of the multiplier used to dualize the allocation constraint at time  $\tau$ . Then the functions  $F_t^i$  and  $G_t^i$  are supermodular<sup>6</sup> in  $(v^i, a_\tau)$  and  $(w^i, a_\tau)$ , respectively, for all  $\tau \geq t$ .*

**LEMMA 6 ( $H_t$  IS SUBMODULAR IN  $(x, a_\tau)$ ).** *Let  $a_\tau$  be the scalar value of the multiplier used to dualize the allocation constraint at time  $\tau$ . The function  $H_t(\mathcal{S}_t, x; \mathbf{a}_t)$  is submodular in  $(x, a_\tau)$  for all  $\tau \geq t$ .*

The properties of the LR-optimal procurement and distribution policy formalized in the theorem below follow directly from Lemmas 5 and 6.

**THEOREM 4 (OPTIMAL POLICY OF  $V_t^L$  IS MONOTONIC IN  $\mathbf{a}$ ).** *The LR-optimal procurement and distribution policy has the following properties:*

(a) *The LR-optimal stocking policy of retailer  $i$  is characterized by a base-stock level  $v^{iu}$ , which is nonincreasing in the value of the multiplier  $a_\tau \geq 0$  for all retailers  $i$  at all time periods  $t \leq \tau$ .*

(b) *The LR-optimal echelon inventory after spot procurement and after forward procurement, denoted respectively as  $\underline{z}^*$  and  $\underline{y}^*$ , are nondecreasing in the value of the multiplier  $a_\tau \geq 0$  for all time periods  $t \leq \tau$ .*

For any given initial inventory  $w^i$  at retailer  $i$ , the LR-optimal inventory at retailer  $i$  after receiving its shipment from the depot is  $\underline{v}^{i*} = \max(w^i, v^{iu})$ , where  $v^{iu}$  is the unconstrained minimizer of (17) and  $\underline{v}_t^* = (\underline{v}^{i*})$ . For any given  $x$  the LR-optimal inventory level after spot procurement is  $\underline{z}^* = \max(x, z^u)$ , and the LR-optimal inventory position after the execution of

<sup>6</sup> A function  $g: \mathbb{R}^n \rightarrow \mathbb{R}$  is supermodular if for any two vectors  $x_1, x_2 \in \mathbb{R}^n$ , we have  $g(x_1 \vee x_2) + g(x_1 \wedge x_2) \geq g(x_1) + g(x_2)$ ; if the inequality is reversed, the function is called submodular (Topkis 1998).

forward procurement is  $y^* = \max(z^*, y^u)$ . Theorem 4 indicates the optimal inventories at the lower echelon (i.e., retailers) and at the upper echelon move in opposite directions in response to changes in the value of the multipliers  $a_i$ ; hence, the value of the multipliers can be used as a lever to find a feasible inventory and distribution policy (i.e., a policy that does not violate the allocation constraint). This is the focus of the next section.

## 4.2. Approximate Procurement and Distribution Policies

Now we develop an approximation  $\tilde{V}_i$  of  $V_i$  using the above decomposition of  $V_i^L$  and the structural results of the optimal policy for Problem  $(V_i)$  presented in Theorem 1. This allows us to obtain an adaptive feasible policy that reflects at any time  $t$  the commodity market prices and the actual inventories at all the locations.

The retailer cost functions,  $G_t^i$ , defined by the dynamic programs specified by Equations (21) and (17), can be solved independently from each other for any given value of the multiplier  $\mathbf{a}$ ; however, these functions do not capture the costs of imbalances in the retailers' inventory holdings. To address this deficiency we rely on precalculations based on the solution of Problem  $(V^L)$  for different values of  $\mathbf{a}$  and make real-time adjustments based on the results obtained in Theorem 1 as new demand, inventory, and price information unfolds as explained below.

We develop the approximation in three steps: First, using (21) and (17) repeatedly we precalculate the base-stock levels for all retailers as a function of the multiplier vector  $\mathbf{a}$ . This precalculated base-stock functions are saved to generate base-stock level for a given  $\mathbf{a}$  and inventory level in a real time. The second step consists of obtaining each retailer's target stocking level based on the retailer's initial inventory and its base stock as a function of the multiplier vector  $\mathbf{a}$ . Then we make preliminary spot procurement decisions and select a preliminary estimate of the multiplier vector  $\mathbf{a}_t$  that reflects the inventory available at the upper echelon as well as the actual inventories at all the retailers. Because this calculation takes place in real time, no tabulation is necessary. Finally, the third step, also performed in real time, consists of making forward procurement decisions, further refining the value of the multiplier vector  $\mathbf{a}_t$  and finalizing the spot procurement level. In §§(4.2.1)–(4.2.3) we elaborate on the algorithmic specifics of these three steps.

**4.2.1. Precalculation of Distribution Base-Stock Levels.** For nonnegative values of the multiplier vector  $\mathbf{a}$ , we solve (21) and (17) for each retailer to obtain the sequence of base-stock vectors  $\mathbf{v}_1^u(\mathbf{a}_1)$ ,  $\mathbf{v}_2^u(\mathbf{a}_2)$ ,  $\dots$ ,  $\mathbf{v}_T^u(\mathbf{a}_T)$ . The elements of each vector  $\mathbf{v}_t^u(\mathbf{a}_t)$

are the base-stock levels of each retailer  $i$  at time  $t$ , denoted as  $v_t^{iu}(\mathbf{a}_t)$ ,  $\mathbf{v}_t^u(\mathbf{a}_t) \equiv (v_t^{iu}(\mathbf{a}_t))$ .

To reduce computational complexity, at any time  $t$ , in our approximation we restrict our attention to vectors  $\mathbf{a}_t$  of the form  $a_t, a, a, \dots, a$  for  $a \geq 0$  (i.e., we assume  $a_{t+1} = \dots = a_T$  is constant and equal to  $a$ ). Notice that this restriction reduces significantly the computational burden to obtain base-stock levels  $v_t^{iu}(\mathbf{a}_t)$  for each retailer  $i$ ; if we consider  $n$  values of  $a_t$  at each time period  $t$ , we will need to calculate only  $n^2$  (instead of  $n^{T-t+1}$ ) base-stock levels for each retailer at each time period. The results in Theorem 4(a) further simplify the computation of  $\mathbf{v}_t^u(\mathbf{a}_t)$  because they imply that at any time  $t$  and for each retailer  $i$ , the base-stock  $v_t^{iu}(\mathbf{a}_t)$  will not increase if either  $a_t$  or  $a_{t+1}$  increases.

**4.2.2. Approximation of Spot Procurement Policies.** At any time  $t$ , once the retailer inventory vector  $\mathbf{w}_t$  is known, we obtain the target stocking level for each retailer  $i$  as  $\hat{v}_t^i(\mathbf{a}_t) = \max(w_t^i, v_t^{iu}(\mathbf{a}_t))$ . Thus, given an initial retailer inventory vector  $\mathbf{w}_t$  and a multiplier vector  $\mathbf{a}_t$ , we can obtain a *feasible* retailer inventory allocation  $\hat{\mathbf{v}}_t(\mathbf{a}_t) \equiv (\hat{v}_t^i(\mathbf{a}_t))$ .

At each period  $t$ , depending on the commodity price state vector  $\mathcal{S}_t$ , we initialize the multiplier sub-vector  $\hat{\mathbf{a}}_t$  by fixing  $\hat{a}_t = \max(\delta_t, 0)$  and  $\hat{a}_{t+1} = \dots = \hat{a}_T = \max(E_{\mathcal{S}_{t+1}|\mathcal{S}_t}[\delta_{t+1}], 0)$ , and we consider the following two cases:

*Case (i).* If  $\sum_{i=1}^N \hat{v}_t^i(\hat{\mathbf{a}}_t) > x$ , then we leave  $\hat{\mathbf{a}}_t$  unchanged and let  $\hat{z}(\hat{\mathbf{a}}_t) = \sum_{i=1}^N \hat{v}_t^i(\hat{\mathbf{a}}_t)$ .

In this case the marginal cost reduction resulting from increasing stocks after all the upper-echelon stock has been allocated (i.e., the imputed shortage cost in Clark and Scarf (1960) terminology) is at least as large as marginal convenience yield  $\delta_t$  for some retailers; and because there is not enough inventory at the upper echelon, procuring from the spot market will lead to a cost reduction. This approximation of  $a_t$  is consistent with the optimal policy behavior, as stated in Theorem 1(a).

*Case (ii).* If  $\sum_{i=1}^N \hat{v}_t^i(\hat{\mathbf{a}}_t) \leq x$ , change the value of  $\hat{a}_t$  to solve  $\sum_{i=1}^N \hat{v}_t^i(\hat{\mathbf{a}}_t) = x$  or set  $\hat{a}_t = 0$  if no positive value of  $\hat{a}_t$  solves it. Set  $\hat{z}(\hat{\mathbf{a}}_t) = x$ .

In this case there is enough inventory at the upper echelon, so no spot procurement is necessary. This is consistent with Theorem 1(b).

**4.2.3. Approximation of Forward Procurement Policies.** Using the value of  $\hat{\mathbf{a}}_t$  obtained above, we now focus on obtaining an approximation  $\tilde{J}_t^2(\mathcal{S}_t, y; \hat{\mathbf{a}}_t)$  of the function  $J_t^2(\mathcal{S}_t, y; \hat{\mathbf{a}}_t)$ , and then we obtain the approximate forward policy  $\tilde{y}_t$  by minimizing  $\tilde{J}_t^2$ . Notice from (16) that  $J_t^2$  requires the computation of  $E_{\zeta_t} E_{\mathcal{S}_{t+1}|\mathcal{S}_t} [H_{t+1}(\mathcal{S}, y - \zeta_t; \hat{\mathbf{a}}_{t+1})]$ ; because this is very complex, we base  $\tilde{J}_t^2$  on an approximation  $\tilde{H}_{t+1}$  of  $H_{t+1}$  obtained using a *limited-look-ahead* approach. We

start our approximation by assuming at time  $t$  that  $t+2$  is the end of the planning horizon; thus we need to define a suitable terminal value function denoted as  $C_{t+2}(\mathcal{S}_{t+2}, x)$ .<sup>7</sup> Using Equations (18)–(20) leads to

$$\begin{aligned} & \tilde{H}_{t+1}(\mathcal{S}_{t+1}, x; \hat{\mathbf{a}}_{t+1}) \\ &= \min_{x \leq z} \left\{ J_{t+1}^1(\mathcal{S}_{t+1}, z; \hat{\mathbf{a}}_{t+1}) + \min_{z \leq y} \{B(\mathcal{S}_{t+1}, y; \hat{\mathbf{a}}_{t+1})\} \right\} \\ & \quad - (s_{t+1} + \eta_{t+1})x, \end{aligned} \quad (22)$$

where

$$\begin{aligned} B(\mathcal{S}_{t+1}, y; \hat{\mathbf{a}}_{t+1}) &= \{\beta(f_{t+1} + \alpha_{t+1})y \\ & \quad + \beta E_{\xi_{t+1}} C_{t+2}(\mathcal{S}_{t+2}, y - \zeta_{t+1})\}. \end{aligned} \quad (23)$$

Define the unconstrained minimizer of the function  $B(\mathcal{S}_{t+1}, y; \hat{\mathbf{a}}_{t+1})$  as  $\hat{y}^u$ . Then  $\tilde{H}_{t+1}$  can be defined as

$$\begin{aligned} & \tilde{H}_{t+1}(\mathcal{S}_{t+1}, x; \hat{\mathbf{a}}_{t+1}) \\ &= \min_{x \leq z} \{J_{t+1}^1(\mathcal{S}_{t+1}, z; \hat{\mathbf{a}}_{t+1}) + B(\mathcal{S}_{t+1}, \hat{y}^u \vee z; \hat{\mathbf{a}}_{t+1})\} \\ & \quad - (s_{t+1} + \eta_{t+1})x, \end{aligned} \quad (24)$$

and based on (16) we define the approximation  $\tilde{J}_t^2$  using a *certainty-equivalent*<sup>8</sup> approach as

$$\begin{aligned} \tilde{J}_t^2(\mathcal{S}_t, y; \hat{\mathbf{a}}_t) &= \beta(f_t + \alpha_t)y \\ & \quad + \beta E_{\xi_t} \tilde{H}_{t+1}(E_{\mathcal{S}|\mathcal{S}_t}^{\mathbb{Q}}[\mathcal{S}], y - \zeta_t; \hat{\mathbf{a}}_{t+1}). \end{aligned} \quad (25)$$

The minimization of the above expression with respect to  $y$  yields the unconstrained minimum  $\tilde{y}^u$  of our approximation. To finalize the specification of our approximate operating policy, we compare the value of  $\tilde{y}^u$  obtained above with  $\sum_{i=1}^N \hat{v}_i^i(\hat{\mathbf{a}}_t)$  for the multiplier vector  $\hat{\mathbf{a}}_t$  obtained in §4.2.2 and consider the following two cases:

*Case (i).* If  $\tilde{y}^u \geq \sum_{i=1}^N \hat{v}_i^i(\hat{\mathbf{a}}_t)$ , then set  $\tilde{y} = \tilde{y}^u$ ,  $\tilde{z} = \hat{z}(\hat{\mathbf{a}}_t)$ ,  $\tilde{\mathbf{v}} = \hat{\mathbf{v}}(\hat{\mathbf{a}}_t)$  to define  $(\tilde{z}, \tilde{y}, \tilde{\mathbf{v}})$  as the approximate policy.

In this case, the policy  $(\tilde{z}, \tilde{y}, \tilde{\mathbf{v}})$  is feasible, and the values of the policy and the multiplier are consistent with Theorem 1(a) and (b).

*Case (ii).* If  $\tilde{y}^u < \sum_{i=1}^N \hat{v}_i^i(\hat{\mathbf{a}}_t)$ , then iterate through Equations (24) and (25) increasing the value of  $\hat{\mathbf{a}}_t$  to a value  $\tilde{\mathbf{a}}_t$  until we find a multiplier subvector  $\hat{\mathbf{a}}_t = (\tilde{\mathbf{a}}_t, \hat{\mathbf{a}}_{t+1})$  that satisfies  $\tilde{y}^u = \hat{z}(\hat{\mathbf{a}}_t)$ , and for the increased value  $\hat{\mathbf{a}}_t$  set  $\tilde{z} = \hat{z}(\hat{\mathbf{a}}_t)$  and  $\tilde{\mathbf{v}} = \hat{\mathbf{v}}(\hat{\mathbf{a}}_t)$ ; define  $(\tilde{z}, \tilde{y}, \tilde{\mathbf{v}})$  as the approximate policy.

<sup>7</sup> Although multiple choices of  $C_{t+2}$  are possible, the specific definition of  $C_{t+2}$  used in our calculations is the cost function of a different approximate model that assumes a deterministic convenience yield. This will be further discussed in §5.3.

<sup>8</sup> By this we mean using  $E_{\xi_t} \tilde{H}_{t+1}(E_{\mathcal{S}|\mathcal{S}_t}^{\mathbb{Q}}[\mathcal{S}], y - \zeta_t; \hat{\mathbf{a}}_{t+1})$  to define  $\tilde{J}_t^2$  instead of  $E_{\xi_t} E_{\mathcal{S}|\mathcal{S}_t}^{\mathbb{Q}}[\tilde{H}_{t+1}(\mathcal{S}, y - \zeta_t; \hat{\mathbf{a}}_{t+1})]$  in (25).

The policy adjustment in this case is representative of instances in which a demand downturn is anticipated, and consistent with Theorem 1(c) we may increase the value of the multiplier  $a_i$  above  $\delta_i$  and reduce (eliminate) spot procurement to match a lower forward procurement inventory position. Theorem 4 parts (a) and (b) imply that the iterative process in Case (ii) above converges. The revised value of the Lagrange multiplier vector  $\hat{\mathbf{a}}_t$ , and hence the approximate policy  $(\tilde{z}, \tilde{y}, \tilde{\mathbf{v}})$ , depend on all future demand distributions  $\Phi_i^i(\cdot), \Phi_{i+1}^i, \dots, \Phi_T^i(\cdot)$  for all retailers  $i$  and of the currently observed price state  $\mathcal{S}_t$ . In the next section, we evaluate numerically the performance of this approximate policy.

We evaluated the lower bound  $V_t^l$  by discretizing the price process state space  $(\chi_t, \omega_t)$  as a binomial lattice to describe the evolution of the commodity price. Then we use this lattice to evaluate the stochastic dynamic program for a given scalar value of  $\lambda$ . We then choose the value of  $\lambda$  that maximizes  $V_t^l$  to obtain the lower bound. The upper bounds obtained through our model are within 3% of the above lower bound in all cases considered in the numerical study.

## 5. Computational Study

In §5.1 we specify a stochastic process to model spot and futures commodity prices. In §5.2 we describe in detail the demand model and various operational parameters considered for numerical computation of various policies through Monte Carlo simulation. In §5.3 we illustrate the four policies evaluated through Monte Carlo simulation and describe the algorithm used to compute them. Then in §5.4 we test the performance of the approximate policies developed in the previous section and develop managerial insights from the sensitivity analysis of various operating parameters on the total inventory cost.

### 5.1. Stochastic Process for the Gasoline Price

To compute the approximate policies developed in §4.2, we need a stochastic process to describe the evolution of the price process  $\mathcal{S}_t$ , and to this end we use the two factor model developed by Schwartz and Smith (2000). This model allows for short-term random variations in prices as well as uncertainty in the long-term level of prices. In this stochastic model, the logarithm of the price at time  $t$ ,  $\ln(s_t)$ , is explained as the combination of two factors: a short-term deviation factor  $\chi_t$  and a long-term equilibrium factor  $\omega_t$ , such that  $\ln(s_t) = \chi_t + \omega_t$ . Thus, in terms of our procurement model notation, the state of the price process,  $\mathcal{S}_t$ , is defined as a vector consisting of the above two factors  $\mathcal{S}_t = (\chi_t, \omega_t)$ . The risk-neutral stochastic process for the short-term deviation factor  $\chi_t$  is the Ornstein–Uhlenbeck process defined as  $d\chi_t = -(\kappa\chi_t + \epsilon)dt + \sigma_\chi dZ$ , where parameter  $\kappa$  is the rate of mean reversion

in the short-term deviations,  $\epsilon$  is the drift reduction and  $\sigma_\chi$  is the volatility associated with the short-term deviation factor. The risk-neutral stochastic process for the long-term factor is assumed to be a Brownian motion process defined as  $d\omega_t = \mu^*dt + \sigma_\omega dW$ ; the parameter  $\mu^*$  is the reduced drift factor, and  $\sigma_\omega$  is the volatility associated with the changes in the long-term equilibrium factor. The terms  $dZ$  and  $dW$  are the increments of two Brownian motions and are related as  $dZ \cdot dW = \rho dt$ , where  $\rho$  is the correlation coefficient between the increments of two stochastic processes. Schwartz and Smith (2000) show that the futures price at time  $t + \Delta t$  is evaluated at time  $t$  as

$$f_{t,t+\Delta t} = E^Q [s_{t+\Delta t}] = A(\Delta t) e^{(e^{-\kappa\Delta t}\chi_t + \omega_t)}, \quad (26)$$

where

$$A(\Delta t) = \exp \left\{ \mu^* \Delta t + (1 - e^{-2\kappa\Delta t}) \frac{\sigma_\chi^2}{4\kappa} + \frac{\sigma_\omega^2}{2} \Delta t + \frac{(1 - e^{-\kappa\Delta t})\rho\sigma_\chi\sigma_\omega}{\kappa} - (1 - e^{-\kappa\Delta t}) \frac{\epsilon}{\kappa} \right\}.$$

## 5.2. Demand Model and Operational Parameters

In our base-case model we consider five identical retailers facing normal demand of 30 barrels a week with a coefficient of variation of 0.167. Later on we consider up to 100 retailers and also analyze the case of nonidentical retailers. To estimate the base-case values of the costs in our distribution model, we consulted a variety of sources. The Office of the Attorney General of the State of Michigan (2009) estimated the cost of distribution from the terminal to the retailing as 4.5 cents per gallon (c/g) on the average across the state. The cost of transportation from the refinery to the terminal varies with the distance between them, and the Association of Oil Pipelines (AOPL 2009) states the cost of shipping petroleum products through a pipeline from the Gulf Coast to New York at 2.5 c/g; other unpublished estimates in the industry range from 1.5 to 2.5 c/g per 1,000 miles. Based on the above information, in our numerical study we selected as base case the transportation costs of gasoline from the terminal to the retailer as  $\gamma_t = 5$  c/g, the forward procurement costs at terminal as  $\alpha_t = 2$  c/g, and the spot procurement at the terminal as  $\eta_t = 2.2$  c/g. In addition, based on the findings of Ashton (2007), we assume the holding cost at the retailer is  $h_t = 1$  cent/gallon/week.

Our analysis uses the stochastic process parameters estimated by Goel (2007) for the prices of gasoline; these parameters were obtained using the Kalman filter technique and the statistical model in Schwartz and Smith (2000) on the daily prices of unleaded gasoline (RBOB/HU contract) traded on NYMEX from October 1999 to May 2005. The values of the parameters obtained for the price process are summarized in Table 1; spot and futures prices of gasoline are stated in cents per gallon.

**Table 1** Summary of Stochastic Process Parameters

Parameter	Symbol	Value
Short-term volatility	$\sigma_\chi$	0.39
Long-term volatility	$\sigma_\omega$	0.16
Short-term mean-reversion rate	$\kappa$	1.63
Correlation in increments	$\rho$	-0.15
Short-term risk premium	$\epsilon$	0.351
Equilibrium risk-neutral drift rate	$\mu^*$	-0.1
Initial conditions		
Short-term factor	$\chi_0$	-0.045
Long-term factor	$\omega_0$	4.3

## 5.3. Algorithm Implementation

The purpose of this numerical study is twofold; first, we want to estimate the value of having the spot market as a faster procurement alternative; second, we want to estimate the value of incorporating the dynamics of commodity price information in the procurement and distribution policy. We use the approximate policies developed in the previous section within a Monte Carlo simulation approach to assess the above benefits on a system consisting of an upper-echelon warehouse and multiple retailers over a 50-week planning horizon; each example is obtained with a combination of 15,000 price paths and 50 demand paths, for a total of 750,000 different paths.

We use four procurement and distribution models each named according to its respective assumptions regarding the evolution of commodity prices and the availability of spot market procurement:

(SS): *Stochastic Convenience Yield with Spot Procurement*. The SS model is the approximation developed in §4.2.

(SN): *Stochastic Convenience Yield with No Spot Procurement*. The SN model is also the approximation developed in §4.2 with the additional constraint that no spot procurement is allowed.

(DS): *Deterministic Convenience Yield with Spot Procurement*. The calculation of the procurement and distribution policy in the DS model assumes the marginal convenience yield, and hence the economic cost of holding, is deterministic. Under the DS model, the futures price curve at time  $t = 0$  is obtained according to the stochastic price process described in §5.1 and denoted as  $s_0, f_{0,1}, f_{0,2}, \dots, f_{0,t}, \dots, f_{0,T}$ . Using these prices we calculate  $\delta_t$  for each  $t$  as  $\delta_t = f_{0,t} + h_t - \beta f_{0,t+1} + \eta_t - \beta \alpha_t$  and then use it to obtain procurement and distribution policies using the algorithm described in §4.2.

(DN): *Deterministic Convenience Yield with No Spot Procurement*. This policy is similar to DS, but no spot procurement is allowed. A detailed description of this model and the DN policy is included in Appendix B. The cost function of this model is used to define the terminal value in the limited-look-ahead approximation used in the SS model.

The relevant inventory costs of the procurement policies associated with using each of these models are denoted as  $C(SS)$ ,  $C(SN)$ ,  $C(DS)$ , and  $C(DN)$  respectively.

The main difference between the  $SS$  (respectively,  $SN$ ) and the  $DS$  (respectively,  $DN$ ) models is that in the  $SS$  (respectively,  $SN$ ) model, the value of the convenience yield is updated at each procurement period throughout the planning horizon. Thus  $C(DS) - C(SS)$  (respectively,  $C(DN) - C(SN)$ ) can be interpreted as the value of up-to-date convenience yield information when spot procurement is available (respectively, when spot procurement is not available). Similarly, the value of the availability of spot procurement is obtained as  $C(SN) - C(SS)$  (respectively,  $C(DN) - C(DS)$ ) when convenience yield information is dynamically updated (respectively, when convenience yield is assumed to be deterministic).

The Schwartz and Smith (2000) price model ensures that there are no arbitrage opportunities in the financial markets, but it cannot prevent arbitrage opportunities in the physical trade because it can conceivably produce spot-futures spreads resulting in a negative marginal convenience yield. This was not a problem for the values of the price parameters we obtained for gasoline, but in general it cannot be ruled out. In our model we addressed this potential problem in two aspects of the model: (1) We set limits on spot procurement up to a maximum inventory position equal to the expected demand plus five standard deviations, and (2) because the Lagrange multiplier of the allocation constraint must be nonnegative, in §4.2.2 we set  $\hat{a}_t = \max(\delta_t, 0)$  and  $\hat{a}_{t+1} = \dots = \hat{a}_T = \max(\mathbb{E}_{\mathcal{F}_{t+1}^Q}[\delta_{t+1}], 0)$  to avoid any potential negative multiplier values.

In our numerical study we compare the total inventory costs.<sup>9</sup> The inventory costs in our model are of the order 1% of the total procurement costs. We base our model comparisons only on inventory-related costs because these are the only costs that can be influenced by the choice of operating policies. Because the profit margins in gasoline distribution are also of the same order of magnitude (Horsley 2007) savings in inventory costs result in increases in profits of the same order of magnitude.

#### 5.4. Numerical Results

We examine the sensitivity of the four models introduced above with respect to operational variables such as penalty cost, coefficient of variation of demand, the difference between spot and forward

procurement transportation costs, and number of retailers; we also examine the sensitivity to commodity price process parameters such as volatility of short-term factor,  $\sigma_\chi$ , and speed of mean reversion  $\kappa$ . For ease of comparison these costs are normalized taking the base case  $C(SS)$  as 100 so that differences can be interpreted as percentages, and they are plotted in Figure 2. Each of the following sensitivities was obtained changing one parameter at a time while maintaining all others at their base-case values.

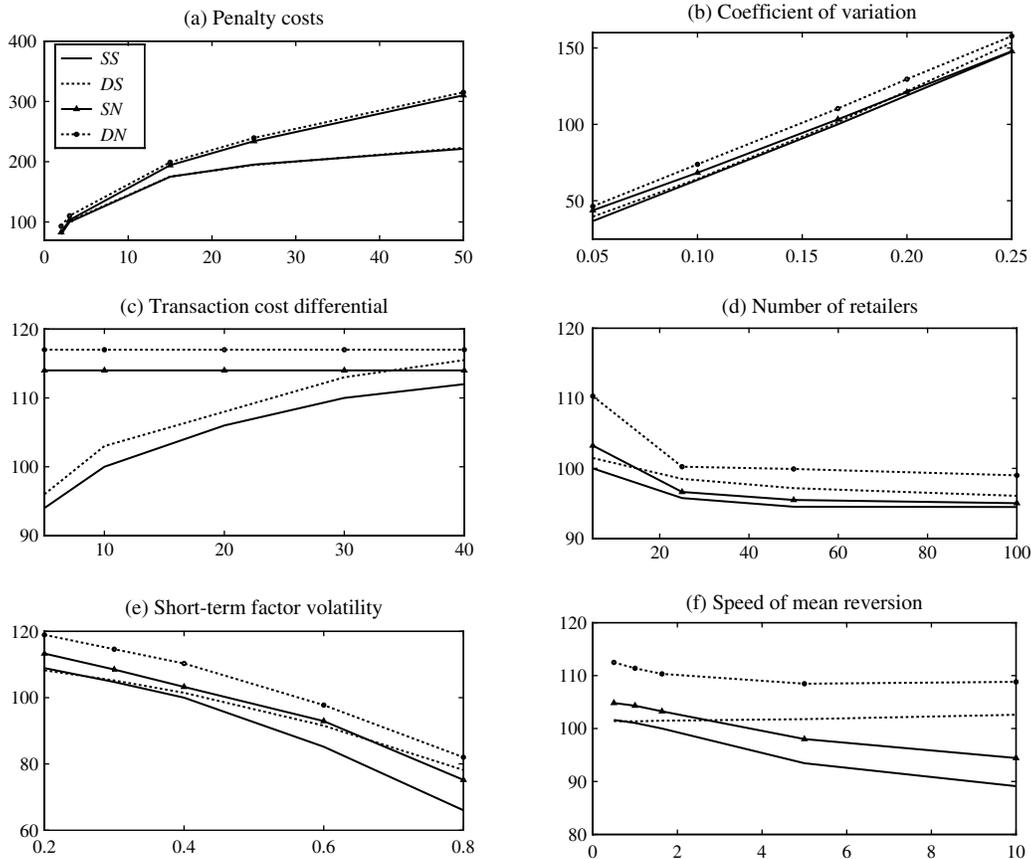
**5.4.1. Effect of Penalty Cost.** We considered penalty cost values ranging from 2 to 50 (cents/gallon-week). Higher penalty costs lead to increased procurement from the spot market because retailers are more willing to trade the higher penalty costs with the holding and transportation costs associated with spot procurement; hence, the value of access to spot market procurement increases with the penalty costs as shown in Figure 2(a). At higher penalty cost the value of dynamic updating of marginal convenience yield is not significant because at higher penalty cost the safety stock level increases, diminishing the impact of changes in convenience yield on the stocking levels.

**5.4.2. Effect of Coefficient of Variation (CV) of Demand.** We considered values of the CV ranging from 0.05 through 0.25 by increasing the value of the standard deviation of demand while keeping the mean of the demand constant. As we increase the value of the CV we continue to see the  $SS$  model dominate any other model. The benefits from  $SS$  model compared to  $DS$  model remain constant in percentage terms, but it increases in absolute terms. As the CV increases, the benefits from dynamically updating the convenience yield information become greater than the flexibility of spot market as can be observed in Figure 2(b) that  $SN$  dominates  $DS$  at higher values of CV. As demand fluctuates so do the inventories in the system, requiring more dynamic updating of the Lagrangian multiplier to ensure efficient allocation of stocks.

**5.4.3. Effect of Spot Transportation Costs Differentials (TCDs).** We considered the TCDs between spot and forward procurement ranging from 5% to 40% of the forward transportation cost. As the TCDs increases, the spot procurement gets more expensive and there is less value in the additional flexibility of spot procurement. We observe in Figure 2(c) that if the TCD is 30% or larger it is better to avoid spot procurement altogether unless spot procurement is done in conjunction with dynamic updating of the marginal convenience yield. Overall, the advantage of dynamically updating the convenience yield information in procurement policy is significant.

<sup>9</sup>These are defined as total costs minus unavoidable costs. The unavoidable costs consist of  $\sum_{i=1}^T \beta^{i+1} \mathbb{E}_{\xi_i} \mathbb{E}_{\xi_i | \xi_0}^Q (f_{i,i+1} + \alpha_{i+1} + \gamma_{i+1}) \xi_i - (s_0 + \eta_0)x_0 - \sum_{j=1}^N \gamma_0^j w_0^j$ , and they represent the minimal procurement and distribution costs of any procurement policy.

Figure 2 Sensitivity of Value of Information and Value of Spot Procurement to Model Parameters



Notes. Base-case parameters:  $r = 5\%$  per year;  $N = 5$ ; mean demand vector = (30, 30, 30, 30, 30); coefficient of variation  $CV^i = 0.167$ ,  $p^i = 3$ , and  $h^i = 1$  for all  $i$ ;  $\eta_t = 2.2$  for all  $t$ ; and  $\alpha = 2$ . The other parameters are as reported in Table 1.

**5.4.4. Effect of Number of Retailers.** We considered the models with the number of retailers ranging from 5 to 100 by keeping the system-wide mean demand and the CV of each retailer constant and assuming the retailers' demands are independent. We observe that as the number of retailers increases, spot procurement becomes more infrequent, and hence the value of the spot procurement decreases. This can be explained as a consequence of demand risk pooling. As the number of retailers increases, the aggregate CV of lead-time demand decreases, and when the larger combined forward orders arrive at the terminal, the likelihood of a retailer inventory unbalance large enough to require spot procurement diminishes; hence, the value of spot market procurement is reduced as the number of retailers increases, as illustrated in Figure 2(d).

**5.4.5. Effect of Short-Term Price Factor Volatility.** We considered values of  $\sigma_\chi$  ranging from 20% to 80% (per year). We observe from Figure 2(e) that as the volatility in the short-term factor increases, the volatility of the marginal convenience yield also increases for the range of parameters considered. This leads to increased value for dynamic information updating in the presence of spot procurement (i.e., the gap

between SS and DS increases); but without access to spot procurement, the benefit of dynamic updating remains roughly constant. At higher short-term factor volatility the SN model dominates the DS model; this elucidates a higher value of dynamic updating of marginal convenience yield than the value of spot procurement flexibility.

**5.4.6. Effect of the Speed of Mean Reversion.** The speed of mean reversion factor,  $\kappa$ , was considered for our experiment the values ranging from 0.5 to 10 ( $\text{years}^{-1}$ ). At lower values of  $\kappa$  the spot and futures prices tend to move in tandem, leading to fewer reversion cycles per unit time and less variance in marginal convenience yield. At higher values of  $\kappa$ , prices revert back to the long-term average faster, thus leading to multiple reversion cycles per year and higher variance in the marginal convenience yield. In Figure 2(f) we can observe that the value of dynamic information updating, both with and without spot procurement, increases as the mean reversion factor  $\kappa$  increases.

**5.4.7. The Case of Nonidentical Retailers.** We considered four configurations, the base case and three unbalanced cases shown in Table 2. The top

**Table 2 Model Comparison for Nonidentical Retailers**

	Balanced (Base case)	Unbalanced $p^i$ (Case I)	Unbalanced CV (Case II)	Unbalanced $p^i$ and CV (Case III)
Penalty cost ( $p^i$ )	(3, 3, 3, 3, 3)	(10, 10, 10, 2, 2)	(3, 3, 3, 3, 3)	(10, 10, 10, 2, 2)
Coef. of variation (CV)	(1/6, ..., 1/6)	(1/6, ..., 1/6)	(0.05, 0.1, 0.15, 0.2, 0.25)	(0.05, 0.1, 0.15, 0.20, 0.25)
$C(SS)^a$	100	100	100	100
$C(DS)$	101	103	103	102
$C(SN)$	103	115	105	126
$C(DN)$	110	121	113	136

<sup>a</sup>Inventory-related costs are normalized to  $C(SS) = 100$  for each case.

panel in Table 2 shows the different vectors of retailers' penalty costs and CV for each of the four cases; thus, a combination of penalty cost and CV vectors represents a different type of retailer imbalance. Case I represent an increasing level of imbalance in penalty costs, Case II represents an imbalance in the CV of demand, and Case III has a combination of both a larger degree of imbalance in CV combined with imbalance in penalty costs. As the imbalance among retailers grows in terms of penalty costs and CV we observe a significant increase in the value of spot procurement because this allows us to re-balance the inventory in the system. On the other hand, the value of dynamic updating of convenience yield information was most significant when the firm has no access to spot procurement ( $C(SN) - C(DN)$ ). The absence of spot procurement increases the importance of dynamic marginal convenience yield information updating because this allows for better allocation of the inventory in the system.

In summary, we can draw the following conclusions from this computational study. The value of spot procurement increases with higher penalty cost and with greater imbalance among retailers. On the other hand, the value of spot procurement decreases as the number of retailers increases due to the effect of demand risk pooling. The value of spot procurement also decreases with higher transaction cost differential as spot procurement becomes expensive. On the other hand, the value of updating convenience yield information increases with higher volatility of the short-term price factor,  $\sigma_{\chi}$ , and with increases in the speed of mean reversion  $\kappa$ . The benefit of updating convenience yield information is also significant when retailers have different penalty costs and spot procurement is unavailable.

## 6. Concluding Remarks and Future Research

In this research, we evaluate the impact of the price information available on commodity markets to enhance the efficacy of the supply chain for traded commodities. The commodity is procured through spot and forward contracts to be further distributed

downstream to  $N$  nonidentical retailers in a two echelon distributive supply chain to satisfy random demand under fluctuating prices. We also assess the value of spot procurement flexibility under various scenarios. We focus on gasoline as an example in our study, but our analysis can be extended to many other traded commodities such as sugar, wheat, heating oil, etc.

Our analysis illustrates that dynamic updating of procurement and distribution policies with market-determined marginal convenience yield information can significantly enhance the efficacy of distributive supply chains when prices are volatile. Therefore, it is important to incorporate price information available on commodity markets in designing the operating policies to achieve lower inventory costs. We develop distribution policies without making a balancing assumption for nonidentical retailers; instead, we develop distribution policies that use the market-determined marginal convenience yield information to obtain the Lagrange multiplier for the allocation constraint.

This area of research, we believe, has promise to advance the theory and managerial practice of procurement and distribution of commodities. There are several possible extensions of this research. One possibility is to relax our assumption of independence in price and demand to study the supply chains of commodities whose demand is sensitive to price. In addition, the assumption of independent demand through time can also be relaxed to evaluate the value of spot procurement flexibility. In this paper, we considered zero lead time from terminal to retailers and spot procurement, and we assumed a lead time of one period for forward procurement. The zero shipping time assumption can be easily relaxed to allow any deterministic positive integer of lead time. In this research, we evaluate the efficacy of a central distribution system; however, an important research extension could be to evaluate decentralized distribution systems.

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## Appendix A. Mathematical Proofs

**PROOF OF LEMMA 1.** Immediate from  $g_1(\mathcal{S}, y - \zeta_t, \mathbf{v} - \vec{\xi}_t) \leq g_2(\mathcal{S}, y - \zeta_t, \mathbf{v} - \vec{\xi}_t)$  and from the observation that the probability distributions in the calculation of  $E^{\mathbb{Q}}$  and  $E_{\vec{\xi}_t}$  are nonnegative.  $\square$

**PROOF OF LEMMA 2.** Because  $g(\mathcal{S}_t, x, \mathbf{w})$  is convex in  $(x, \mathbf{w})$ , then  $g(\mathcal{S}, y - \zeta_t, \mathbf{v} - \vec{\xi}_t)$  is convex in  $(y, \mathbf{v})$  for every demand outcome  $\vec{\xi}_t$ , and every outcome of the price state vector  $\mathcal{S}$  in period  $t + 1$ . Because all probabilities are nonnegative,  $E_{\vec{\xi}_t} E_{\mathcal{S} | \mathcal{S}_t}^{\mathbb{Q}} [f(\mathcal{S}, y - \zeta_t, \mathbf{v} - \vec{\xi}_t)]$  is convex in  $(y, \mathbf{v})$  for every  $\mathcal{S}_t$ . The result follows because the rest of the terms in the definition of  $\mathcal{H}_t$  are convex.

**PROOF OF LEMMA 3.** Using backward induction assume that  $V_{t+1}$  is convex in  $(x, \mathbf{w})$ . This assumption can easily be verified for  $V_{T+1}$ . Moreover, if  $V_{t+1}$  is convex, it follows from Lemma 2 that  $\mathcal{H}_t[V_{t+1}]$  is convex in  $(x, y, z, \mathbf{w}, \mathbf{v})$  for every  $\mathcal{S}_t$ ; and because the minimization in (3) is taken over a convex set, it follows that  $V_t$  is convex in  $(x, \mathbf{w})$ .  $\square$

**PROOF OF THEOREM 1.** Immediate from (4), (7), and (9).  $\square$

**PROOF OF PROPOSITION 1.** Immediate from (7), (4), and (8).  $\square$

**PROOF OF THEOREM 2.** We proceed by induction by assuming  $V_{t+1}^L \leq V_{t+1}$ . It can be verified that this assumption is true in period  $T + 1$ . Then we can observe that

$$\begin{aligned} V_t^L(\mathcal{S}_t, x, \mathbf{w}) &= \min_{\substack{x \leq z \leq y \\ w^i \leq v^i}} \mathcal{H}_t^L[V_{t+1}^L] \leq \min_{\substack{x \leq z \leq y \\ w^i \leq v^i}} \mathcal{H}_t^L[V_{t+1}] \\ &\leq \min_{\substack{x \leq z \leq y \\ w^i \leq v^i \\ z \geq \sum_{i=1}^N v^i}} \mathcal{H}_t[V_{t+1}] = V_t(\mathcal{S}_t, x, \mathbf{w}). \end{aligned} \quad (\text{A1})$$

The first inequality follows from the induction assumption (i.e.,  $V_{t+1}^L \leq V_{t+1}$ ) because  $\mathcal{H}_t^L[V_{t+1}^L] \leq \mathcal{H}_t^L[V_{t+1}]$  pointwise as  $\mathcal{H}_t^L$  is isotonic (as a consequence of Lemma 1). The second inequality follows from the definition of  $\mathcal{H}_t^L$  because the program in the left is a relaxation of the program in the right and  $V_{t+1}$  is independent of  $\mathbf{a}$ .  $\square$

**PROOF OF LEMMA 5.** We proceed using backward induction on  $t$ . First observe that the functions  $G_{\tau'}^i$  and  $F_{\tau'}^i$  are trivially supermodular in  $(w, a_{\tau'})$  and  $(v, a_{\tau'})$ , respectively, for  $\tau' > \tau$  because they do not depend on  $a_{\tau}$ . Next we consider separately the cases of  $F_{\tau}^i(v^i; \mathbf{a}_{\tau})$  and  $F_t^i(v^i; \mathbf{a}_t)$ ,  $t < \tau$ , as defined in (17). Because  $G_{\tau+1}^i$  is independent from  $a_{\tau}$ , and the product  $a_{\tau} v^i$  is supermodular in  $(v^i, a_{\tau})$ , it follows that  $F_{\tau}^i(v^i; \mathbf{a}_{\tau})$  is supermodular in  $(v^i, a_{\tau})$ . Now assume that  $G_{t+1}^i(w; \mathbf{a}_{t+1})$ ,  $t + 1 \leq \tau$ , is supermodular in  $(w, a_{\tau})$ . Because the class of supermodular functions is closed under addition and positive scalar multiplication, it follows that the expectation  $E_{\xi_t} G_{t+1}^i(v^i - \xi_t^i; \mathbf{a}_{t+1})$  is supermodular in  $(v^i, a_{\tau})$ , and  $F_t^i(v^i; \mathbf{a}_t)$  is supermodular in  $(v^i, a_{\tau})$  for  $t < \tau$ . Because the minimization constraint  $w^i \leq v^i$  in (21) is a sublattice

of  $\mathbb{R}^2$ , it follows from Theorem 4.3 of Topkis (1978) that  $G_t^i$  is supermodular in  $(w, a_{\tau})$ .  $\square$

**PROOF OF LEMMA 6.** Using backward induction on  $t$ , we first observe from the definition of  $H_{T+1}$  in the proof of Theorem 3 that  $H_{T+1}$  is trivially submodular in  $(x, a_{\tau})$ . Next we assume for an arbitrary  $t$  that  $H_{t+1}$  is submodular. Because the class of submodular functions is closed under addition and positive scalar multiplication, it follows that  $E_{\xi_t} E_{\mathcal{S} | \mathcal{S}_t}^{\mathbb{Q}} [H_{t+1}(\mathcal{S}, y - \zeta_t; \mathbf{a}_{t+1})]$  is submodular in  $(y, a_{\tau})$ , and it follows from (16) that  $J_t^2(\mathcal{S}_t, y; \mathbf{a}_{t+1})$  is submodular in  $(y, a_{\tau})$ . Moreover, because the product  $-a_{\tau} z$  is submodular, it is immediate from (15) that  $J_t^1(\mathcal{S}_t, z; a_t)$  is submodular in  $(z, a_{\tau})$  for  $t = \tau$  and trivially submodular for  $t \neq \tau$ ; hence,  $J_t^2(\mathcal{S}_t, y; \mathbf{a}_{t+1}) + J_t^1(\mathcal{S}_t, z; a_t)$  is submodular in  $(y, z, a_{\tau})$ . It follows from (18)–(20) that  $H_t$  can be expressed as

$$H_t(\mathcal{S}_t, x; \mathbf{a}_t) = \min_{x \leq z \leq y} \{J_t^1(\mathcal{S}_t, z; a_t) + J_t^2(\mathcal{S}_t, y; \mathbf{a}_{t+1})\}, \quad (\text{A2})$$

and because the constraint  $x \leq z \leq y$  defines a polyhedral sublattice of  $\mathbb{R}^3$ , Theorem 4.3 of Topkis (1978) implies that  $H_t$  is submodular in  $(x, a_{\tau})$ .  $\square$

**PROOF OF THEOREM 4.** The base-stock form for the retailers stocking policies claimed in (a) follows from the convexity in  $w^i$  of  $G_t^i(w^i; \mathbf{a}_t)$  for any  $\mathbf{a}_t \geq 0$ . From Lemma 5 we know  $F_t^i(v^i; \mathbf{a}_t)$  is supermodular in  $(v^i, a_{\tau})$ ,  $\tau \geq t$ , and because the minimization constraint  $w^i \leq v^i$  in (21) is a sublattice of  $\mathbb{R}^2$ , it follows from Theorem 6.1 of Topkis (1978) that  $v^{iu}$  is nonincreasing in  $a_{\tau}$ . The monotonicity properties claimed in (b) follow because  $\underline{z}^*$  and  $\underline{y}^*$  are obtained from the minimization of a submodular function, subject to a constraint set characterized as a sublattice of  $\mathbb{R}^3$ , as defined in (A2) (see proof of Lemma 6); hence, Theorem 6.1 of Topkis (1978) implies both  $\underline{z}^*$  and  $\underline{y}^*$  are nondecreasing in  $a_{\tau}$ .  $\square$

## Appendix B. Model DN: Stochastic Convenience Yield with No Spot Procurement

In this model there is no possibility of spot procurement and the forward procurement policy is obtained using the futures prices observed at time  $t = 0$ , denoted as  $f_{0,t}$ , and assuming they remain constant thereafter; in the DN model the  $f_{0,t}$  are deterministic model parameters. Let  $U_t$  denote the optimal procurement cost under the above assumptions. Using similar notational conventions as in §3,  $U_t$  can be defined as

$$\begin{aligned} U_t(x_t, \mathbf{w}_t) &= \min_{\substack{w^i \leq v^i \\ \sum_{j=1}^N v^j \leq x_t \leq y_t}} \left\{ \beta(f_{0,t} + \alpha_t)(y_t - x_t) + \sum_{j=1}^N \gamma_t^j (v_t^j - w_t^j) \right. \\ &\quad \left. + \sum_{j=1}^N L^j(v^j) + h \left( x_t - \sum_{j=1}^N v_t^j \right) + \beta E_{\xi_t} U_{t+1}(y - \zeta_t, \mathbf{v} - \vec{\xi}_t) \right\}. \end{aligned} \quad (\text{B1})$$

We now dualize the constraint  $\sum_{j=1}^N v^j \leq x_t$  and introduce the Lagrange multiplier  $\lambda_t$  corresponding to this constraint. As in §4, this allows us to obtain a lower bound for

$U_t^l \leq U_t$  that is separable in  $(x, \mathbf{w}_t)$  as  $U_t^l(x_t, \mathbf{w}_t) = M_t(x_t) + \sum_{j=1}^N R_t^j(w_t^j)$ , where

$$M_t(x_t) = \min_{y_t \geq x_t} \left\{ \beta(f_{0,t} + \alpha_t)(y_t - x_t) + (h - \lambda_t)x_t + \beta E_{\xi_t} M_{t+1}(y - \xi_t) \right\}, \quad (B2)$$

$$R_t^j(w_t^j) = \min_{v_t^j \geq w_t^j} \left\{ \gamma_t^j(v_t^j - w_t^j) + \sum_{j=1}^N L^j(v^j) + (\lambda_t - h)v_t^j + \beta E_{\xi_t^j} R_{t+1}^j(v_t^j - \xi_t^j) \right\}. \quad (B3)$$

From the first-order condition for  $y_t$  obtained from (B2) we get

$$\lambda_{t+1} \approx (f_{0,t} + \alpha_t) + h - \beta(f_{0,t+1} + \alpha_{t+1}). \quad (B4)$$

Similarly, for each location  $j$ , from the first-order conditions for  $v_t^j$  derived from (B3) we obtain

$$\lambda_t \approx p^j - (p^j + h^j)\Phi_t^j(v_t^j) + h - \gamma_t^j + \beta\gamma_{t+1}^j. \quad (B5)$$

Assuming demand for each retailer  $j$  is normally distributed with mean  $\mu_t^j$  and variance  $(\sigma_t^j)^2$ , define the aggregate variables  $\mu_t = \sum_j \mu_t^j$  and  $\tilde{\sigma}_t = \sum_j \sigma_t^j$ . Further, assume  $\gamma_t^j = \gamma^j$  for all  $t$  and define  $p^s = (\sum_j p^j)/N$ ,  $h^s = (\sum_j h^j)/N$ , and  $\gamma = (\sum_j \gamma^j)/N$ . Following an aggregation approach similar to the one used in Federgruen and Zipkin (1984a) we get  $\sum_{j=1}^N v_t^j = x_t = \sum_{j=1}^N \mu_t^j + \sum_{j=1}^N \sigma_t^j \Phi_t^{-1}((p^s + h - \hat{\lambda} - (1 - \beta)\gamma)/(p^s + h^s))$ , and we can approximate (B5) as

$$\lambda_t(x_t) \approx (p^s + h) - (p^s + h^s)\Phi_t\left[\frac{x_t - \mu_t}{\tilde{\sigma}_t}\right] - (1 - \beta)\gamma. \quad (B6)$$

Because  $x_{t+1} = y_t - \xi_t$ , at time  $t$  we can use (B6) to define the random variable  $\tilde{\lambda}_{t+1}(y_t - \xi_t)$ . By setting  $E_{\xi_t} \tilde{\lambda}_{t+1}(y_t - \xi_t) = \lambda_{t+1}$  we obtain from (B4) and (B6)

$$E_{\xi_t} \Phi_t\left[\frac{y_t - \xi_t - \mu_t}{\tilde{\sigma}_t}\right] \approx \frac{p^s + (1 - \beta)\gamma - (f_{0,t+1} + \alpha_{t+1}) + (f_{0,t+2} + \alpha_{t+2})}{p^s + h^s}. \quad (B7)$$

The optimal  $y_t$  for the DN model is obtained by solving the above approximate equation.

In §5.4 we also used this model to generate boundary conditions for the limited-look-ahead policies of the SS model by setting  $C_{t+2} = M_{t+2}$ .

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