

A Discrete Time Approach for Modeling Two-Factor Mean-Reverting Stochastic Processes

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Two-factor stochastic processes have been developed to more accurately describe the intertemporal dynamics of variables such as commodity prices. In this paper we develop an approach for modeling these types of stochastic processes in discrete time as two-dimensional binomial sequences. This approach facilitates the numerical solution of dynamic optimization problems such as investment decision making under uncertainty and option valuation related to commodities. We implement this approach in a two-dimensional lattice format, apply it to two hypothetical valuation problems discussed by Schwartz and Smith, and compare the results to those from simulation- and dynamic-programming-based methods.

Key words: stochastic processes; discrete models; dynamic programming; valuation

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1. Introduction

Important classes of dynamic optimization problems can be modeled based on the evolution of uncertainty in the state variable over time as a stochastic process, including problems that address the valuation of financial and real options. Various approaches to stochastic process modeling and option valuation have been developed in the fields of financial engineering and computational economics, which might also be applied to the valuation of some real options problems. Some examples include using a partial differential equation approach based on extensions of the classic work of Black and Scholes (1973), the determination of approximate but explicit solutions based on perturbation methods (e.g., Hikspoors and Jaimungal 2008), Fourier transform methods using Fourier time-stepping approaches (Jaimungal and Surkov 2008), dynamic programming methods using continuous approximations (Reiter 1999), and enhanced solution strategies applied to methods of stochastic control (Chockalingam and Muthuraman 2011). Although these methods do offer the promise of computational advantages and analytical insights in some applications, they may also require a high

degree of sophistication and significant development work in the context of the problem complexity associated with some real option problems.

Our focus in this paper is on lattice-based methods and closely related numerical techniques that provide a discrete approximation of the stochastic process. Lattice-based methods are well established as a tool for valuing options, perhaps because of their ease of implementation and their intuitive appeal. In addition, these lattice models may be especially attractive for members of the decision analysis community because they have an equivalent representation as binomial probability trees and can therefore be developed with commercially accessible decision tree software (Brandão et al. 2005a, b).

Several approaches have been suggested for determining a discrete approximation of an underlying stochastic process. The first example of this approach was a binomial lattice model that converges weakly to a geometric Brownian motion diffusion process, or GBM (Cox et al. 1979). This binomial model can be used to accurately approximate solutions from the Black–Scholes–Merton (Black and Scholes 1973) continuous-time option valuation model. Moreover,

this approach can also be used to solve for the value of early-exercise American options, whereas the Black–Scholes–Merton model can only value European options.

The assumption of a log-normal geometric Brownian diffusion as a model of the underlying stochastic process may not be valid for many option valuation problems, however. An example of such a problem is an option on a real asset with cash flows that depend on a mean-reverting commodity price. Several empirical studies of historical data (e.g., Schwartz 1997) and futures data (e.g., Bessembinder et al. 1995) have found that mean-reverting processes are often the most accurate models for capturing the behavior of commodity prices. Moreover, if commodity prices are indeed mean reverting, then a log-normal geometric Brownian diffusion model can significantly overestimate uncertainty in the resultant cash flows from a project and result in overstated option values, as noted by Schwartz (1998), Loughton and Jacoby (1993), and others.

Relative to geometric Brownian processes, however, discrete time modeling of mean-reverting stochastic processes for use in option valuation models has proven to be much more difficult. Methods employing Monte Carlo simulation and discrete trinomial lattices have been the primary proposed approaches for this purpose. Monte Carlo methods such as the approach of Longstaff and Schwartz (2001) have the advantage of being able to accommodate general types of stochastic processes and can also be used to value early-exercise options, a shortcoming of traditional simulation-based methods. Their method uses *ex post* regression of cash flows on state values at each step to estimate the value function used to determine the optimal stopping rule and the option value, as illustrated, for example, by Smith (2005). Unfortunately, a significant drawback of this approach is that it is computationally intensive, particularly for problems with multiple concurrent options. Implementations that take advantage of parallel computing might mitigate some of these concerns and be especially worthwhile in situations involving financial options where the analysis might be repeated many times. However, this would add to the algorithmic complexity of the implementation, which might be less appealing for one-time applications to unique real options problems.

Hull and White (1993), Tseng and Lin (2007), and others have provided examples of discrete trinomial and multinomial lattices for valuing options in a similar manner to the binomial approach, but with the ability to model more general types of stochastic processes because of the additional degrees of freedom. Unfortunately, these types of lattices require more involved methodologies for specifying valid branching probabilities and lattice cell sizes to ensure convergence of the stochastic process, and their application is also limited in multifactor cases by an upper bound on the correlation between factors.

As an alternative, we have developed a method, based on the approach proposed by Nelson and Ramaswamy (1990), for constructing binomial models for homoskedastic mean-reverting stochastic processes that can be used to model correlated, one-factor, mean-reverting stochastic processes, such as the price of oil and the price of natural gas (Hahn and Dyer 2008). In this paper, we extend this approach to the two-factor process developed by Schwartz and Smith (2000) to model the dynamics of the long-term equilibrium level along with short-term deviations from the equilibrium level and then provide a computational comparison with the other alternatives for the particular case of modeling two-factor processes.

This paper is organized as follows: In §2, we review the Nelson and Ramaswamy (1990) approach to constructing computationally simple binomial lattices and apply the approach to a one-factor mean-reverting process. In §3, we develop a binomial representation of the two-factor diffusion model of Schwartz and Smith (2000). Section 4 details an application of the two-factor model to solve two sample real options problems and to compare results to those from a simulation-regression model and a grid-based dynamic programming model. We conclude in §5 with a summary and discussion of key findings.

2. General Method of Developing Recombining Lattices

Cox et al. (1979) developed a method for specifying parameters for a recombining binomial lattice based on the stochastic differential equation (SDE) $dS/S = \mu dt + \sigma dz$, which represents a GBM model of

the diffusion of an asset price over time. The Ito-transformed process for the log of asset price is

$$d \ln S = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dz, \tag{1}$$

where S is the asset price (e.g., stock price), μ is the growth rate (drift), σ is the standard deviation of returns (volatility), and dz is a Wiener process.

This binomial model has been very popular in valuing financial options and many types of real options where a GBM model is a reasonable representation of the diffusion of the underlying asset value. The binomial model also has the important computational property of recombination; that is, branches of the binomial lattice reconnect at each step. This facilitates the solution of very large problems because there are $N + 1$ nodes at any stage N , whereas there are 2^N nodes at the same stage for a binary tree.

In cases where the underlying stochastic process cannot be accurately modeled as a GBM model unless a transformation is applied (see K llezi and Webber 2004), it is useful to have a modeling procedure similar to that of Cox et al. (1979) that exhibits convergence in distribution and is yet computationally simple and robust in terms of allowable payoff specifications. Nelson and Ramaswamy (1990) propose an approach that meets these requirements under very general conditions. In that case, the problem is to find a binomial sequence that converges to an SDE of the general form

$$dY_t = \mu(Y, t)dt + \sigma(Y, t) dz, \tag{2}$$

where $\mu(Y, t)$ and $\sigma(Y, t)$ are continuous instantaneous drift and standard deviation functions, respectively, and dz is a standard Brownian increment. To solve this problem, Nelson and Ramaswamy (1990) propose a simple binomial sequence of n periods of length h , where $h = T/n$, and

$$Y_h^+ \equiv Y + \sqrt{h}\sigma(Y, t) \quad (\text{up move}), \tag{3}$$

$$Y_h^- \equiv Y - \sqrt{h}\sigma(Y, t) \quad (\text{down move}), \tag{4}$$

$$q_h \equiv \frac{1}{2} + \sqrt{h} \frac{\mu(Y, t)}{2\sigma(Y, t)} \quad (\text{probability of up move}), \tag{5}$$

$$1 - q_h \quad (\text{probability of down move}). \tag{6}$$

For diffusions with constant variance, this approach thus entails fixing the up and down moves in the lat-

tice and calculating probabilities at each node, conditioned on the state, to reflect the local drift. In any cases where nodes have invalid branching probabilities, the probabilities are censored so that negative probabilities are set to zero and probabilities greater than one are set to one.

Nelson and Ramaswamy (1990) show that as the time increment is reduced, the drift and variance of this approximation converge to those of the continuous diffusion. More formally, the conditions under which this sequence converges to the above SDE are (i) that the SDE is well behaved (i.e., that $Y_t = Y_0 + \int_0^t \mu(Y_s, s) ds + \int_0^t \sigma(Y_s, s) dz_s$ exists on $0 < t < \infty$) and (ii) that the jump sizes, local drift, and local variance converge in distribution (i.e., $|Y_h^\pm(Y, t) - Y|$, $|\mu_h(Y, t) - \mu(Y, t)|$, and $|\sigma_h^2(Y, t) - \sigma^2(Y, t)| \rightarrow 0$ as $h \rightarrow 0$).

This method can be used to develop discrete models of mean-reverting processes that have the same desirable properties of discrete GBM models. The simplest case is a one-factor mean-reverting process, the Ornstein-Uhlenbeck process, which is given by

$$dY_t = \kappa(\bar{Y} - Y_t) dt + \sigma dz_t, \tag{7}$$

where Y_t is the log of commodity price, κ is a mean-reversion coefficient, \bar{Y} is the log of the long-term equilibrium price, σ is the process volatility, and dz is a Wiener process. The log is used because it is commonly assumed that commodity prices are log-normally distributed.

Substituting $\kappa(\bar{Y} - Y_t)$ for $\mu(Y, t)$ and σ for $\sigma(Y, t)$ in the general binomial sequence shown in Equations (3)–(6) yields the following binomial model of a one-factor mean-reverting process:

$$Y_h^+ \equiv Y + \sqrt{h}\sigma \quad (\text{up move}); \tag{8}$$

$$Y_h^- \equiv Y - \sqrt{h}\sigma \quad (\text{down move}); \tag{9}$$

$$q_h \equiv \begin{cases} \frac{1}{2} + \sqrt{h} \frac{\kappa(\bar{Y} - Y_t)}{2\sigma} & \text{if } 0 \leq \frac{1}{2} + \sqrt{h} \frac{\kappa(\bar{Y} - Y_t)}{2\sigma} \leq 1, \\ 0 & \text{if } \frac{1}{2} + \sqrt{h} \frac{\kappa(\bar{Y} - Y_t)}{2\sigma} \leq 0, \\ 1 & \text{if } 1 \leq \frac{1}{2} + \sqrt{h} \frac{\kappa(\bar{Y} - Y_t)}{2\sigma} \end{cases}$$

(probability of up move); (10)

$$1 - q_h \quad (\text{probability of down move}). \tag{11}$$

A key aspect of this model is the censoring of transition probabilities to values between 0 and 1 in Equation (10), which can be rewritten in a single statement as

$$q_h = \max\left(0, \min\left(1, \left(\frac{1}{2} + \sqrt{h} \frac{\kappa(\bar{Y} - Y_t)}{2\sigma}\right)\right)\right). \quad (12)$$

3. Modeling a Two-Factor Mean-Reverting Process

Although the one-factor model in the previous section can be used to capture mean reversion in a variable such as commodity price, it assumes there is no uncertainty in the long-term equilibrium level. Schwartz and Smith (2000), Gibson and Schwartz (1990), and others have introduced composite diffusions that include a second factor to explicitly model uncertainty in the short-term deviations from the long-term equilibrium level, as well as in the equilibrium level itself. Binomial approximations of two-factor diffusions were first introduced by Boyle (1988) and have been used extensively in modeling interest rate dynamics. He (1990), Ho et al. (1995), and others have demonstrated that the convergence of these models duplicates the univariate case for GBM diffusions. Jaillet et al. (2004) presented an approach to modeling multiple one-factor mean-reverting processes, albeit for the same underlying asset, in the context of valuing swing options.

Hull and White (1994) propose a dual trinomial lattice approach for modeling two-factor mean-reverting diffusions that is an extension of their earlier work on one-factor models. As pointed out by Tseng and Lin (2007), however, this approach is limited by the requirement of a heuristic method for determining process increments and branching probabilities and can only guarantee a feasible lattice if the correlation between factors is less than 0.2. Tseng and Lin (2007) improve upon the Hull and White (1994) two-factor model by introducing an optimization approach to determining the feasible lattice structure that offers the best convergence behavior; however, this adds complexity to the trinomial approach and still only guarantees a feasible lattice if the correlation between factors is no more than 0.676. Therefore, this approach would not apply to many problems, including, for example, a switching option between outputs of a

sugarcane processing facility that can produce ethanol or sugar, where the correlation between price processes was found to be 0.78 (Bastian-Pinto et al. 2009). We show that the general approach for binomial lattice construction can be extended to the case of two-factor models so that one or both of the factors can follow a mean-reverting diffusion, without limitation to particular grid sizes, branching schemes, or range of process parameter values.

In the Schwartz and Smith (2000) model, the logarithm of the price at any point is decomposed into two factors: a long-term equilibrium price, ξ_t , and a deviation from the equilibrium price, χ_t . The long-term equilibrium price is specified to follow a Brownian motion, whereas the short-term deviation follows a simple one-factor Ornstein–Uhlenbeck process, and eventually reverts to zero. The price is therefore given by

$$y_t = e^{\chi_t + \xi_t}, \quad (13)$$

where $d\xi_t = \mu_\xi dt + \sigma_\xi dz_\xi$ is the process for long-term equilibrium price, and $d\chi_t = \kappa(\bar{\chi} - \chi_t) dt + \sigma_\chi dz_\chi$ is the process for the deviation from the long-term price. The relationship between the increments of the two processes is given by $dz_\xi dz_\chi = \rho_{\xi\chi} dt$.

A two-dimensional binomial approximation can be developed for this process, which results in a four-branch node for each discrete period, as shown in Figure 1.

To define the joint probabilities for the node shown in Figure 2, the state-space increments for each factor can first be set to

$$\Delta_\xi = \sigma_\xi \sqrt{\Delta t}, \quad (14)$$

$$\Delta_\chi = \sigma_\chi \sqrt{\Delta t}, \quad (15)$$

Figure 1 Four-Branch Node for Two-Factor Process

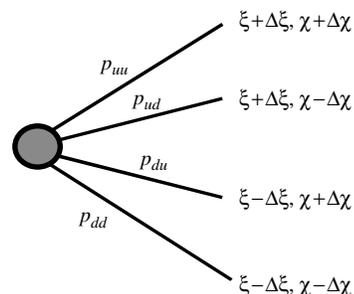
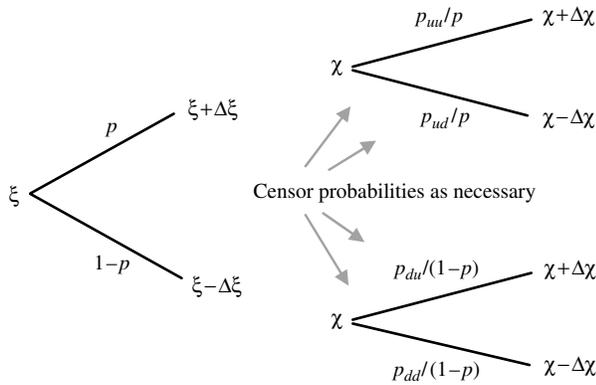


Figure 2 Splitting the Four-Branched Node Into Marginal and Conditional Steps



and, for simplicity, we denote the drifts of the Brownian motion process for ξ and the Ornstein-Uhlenbeck process for χ as

$$v_\xi = \mu_\xi, \tag{16}$$

$$v_\chi = \kappa(\bar{\chi} - \chi_t). \tag{17}$$

Then, using the same basic method employed by Boyle (1988) for a dual log-normal approximation, the probabilities of the four possible combined outcomes can be obtained by matching the mean and variance of a two-variable binomial process:

$$p_{uu} = \frac{\Delta_\xi \Delta_\chi + \Delta_\chi v_\xi \Delta t + \Delta_\xi v_\chi \Delta t + \rho \sigma_\xi \sigma_\chi \Delta t}{4 \Delta_\xi \Delta_\chi}, \tag{18}$$

$$p_{ud} = \frac{\Delta_\xi \Delta_\chi + \Delta_\chi v_\xi \Delta t - \Delta_\xi v_\chi \Delta t - \rho \sigma_\xi \sigma_\chi \Delta t}{4 \Delta_\xi \Delta_\chi}, \tag{19}$$

$$p_{du} = \frac{\Delta_\xi \Delta_\chi - \Delta_\chi v_\xi \Delta t + \Delta_\xi v_\chi \Delta t - \rho \sigma_\xi \sigma_\chi \Delta t}{4 \Delta_\xi \Delta_\chi}, \tag{20}$$

$$p_{dd} = \frac{\Delta_\xi \Delta_\chi - \Delta_\chi v_\xi \Delta t - \Delta_\xi v_\chi \Delta t + \rho \sigma_\xi \sigma_\chi \Delta t}{4 \Delta_\xi \Delta_\chi}. \tag{21}$$

This model combines two recombining lattices, and therefore it is also recombining. Because the model is again based on the Nelson and Ramaswami (1990) general binomial approximation, it may be necessary to censor probabilities when the amount of mean reversion induced at a particular state for the deviation from the long-term mean results in probabilities greater than one or less than zero in any binomial node. Unfortunately, for a four-branch node for a joint

process, it is not possible to directly censor the probabilities as described earlier for the one-factor process. Therefore, we must revise our approach so that it will again satisfy the constraints for modeling the mean-reverting process with a binomial branching scheme while also accounting for possible correlation between the two processes.

The solution to this problem is to represent the joint process as the product of the marginal binomial process for ξ and the conditional binomial process for χ :

$$p(\xi_t, \chi_t) = p(\chi_t | \xi_t) p(\xi_t). \tag{22}$$

We must therefore derive the expressions for the conditional probabilities for χ . To do this, we divide the two joint probabilities that involve an upward move in equilibrium level (p_{uu} and p_{ud}) by the marginal probability of an up move in equilibrium level

$$p_u = \frac{1}{2} + \frac{1}{2} \frac{v_\xi \Delta t}{\Delta_\xi}, \tag{23}$$

and the two joint probabilities that involve a downward move in equilibrium level (p_{du} and p_{dd}) by the marginal probability of a down move in equilibrium level $p_d = 1 - p_u$.

This yields the following two pairs of conditional probabilities:

$$p_{u|u} = \frac{\Delta_\xi(\Delta_\chi + \Delta t v_\chi) + \Delta t(\Delta_\chi v_\xi + \rho \sigma_\xi \sigma_\chi)}{2 \Delta_\chi(\Delta_\xi + \Delta t v_\xi)}, \tag{24}$$

$$p_{d|u} = \frac{\Delta_\xi(\Delta_\chi - \Delta t v_\chi) + \Delta t(\Delta_\chi v_\xi - \rho \sigma_\xi \sigma_\chi)}{2 \Delta_\chi(\Delta_\xi + \Delta t v_\xi)}, \tag{25}$$

and

$$p_{u|d} = \frac{\Delta_\xi(\Delta_\chi - \Delta t v_\chi) + \Delta t(\rho \sigma_\xi \sigma_\chi - \Delta_\chi v_\xi)}{2 \Delta_\chi(\Delta_\xi + \Delta t v_\xi)}, \tag{26}$$

$$p_{d|d} = \frac{\Delta_\xi(\Delta_\chi + \Delta t v_\chi) - \Delta t(\Delta_\chi v_\xi + \rho \sigma_\xi \sigma_\chi)}{2 \Delta_\chi(\Delta_\xi + \Delta t v_\xi)}. \tag{27}$$

The resulting discrete model, shown in Figure 2, is then a two-node sequence: a binomial node for the Brownian motion process for ξ , followed by binomial nodes for the conditional process for χ , which can now be censored as necessary. Proof of convergence for this discrete model to the continuous two-factor diffusion is shown in the appendix.

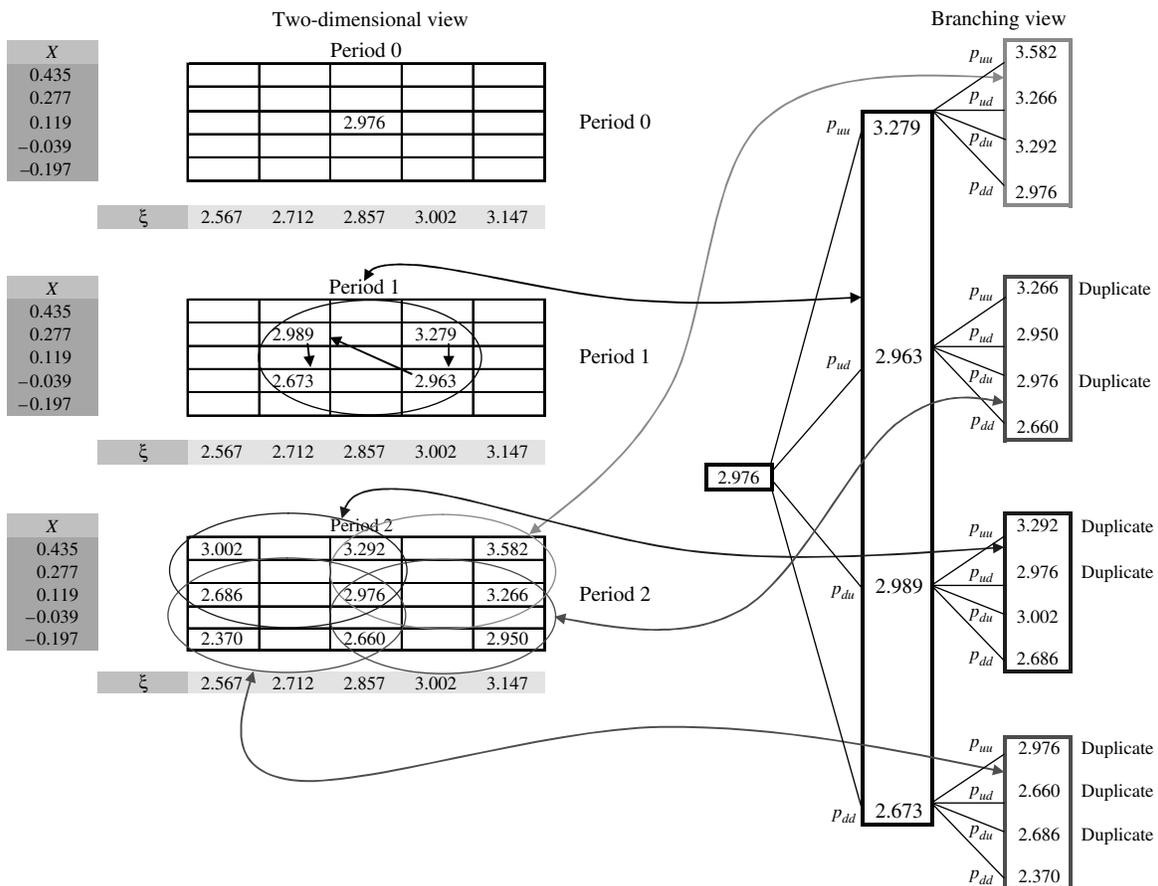
Implementation of this approach is straightforward in a decision tree format using software that will

allow node probabilities to be conditioned on prior nodes in a tree. However, because a decision tree with this configuration would have 4^N endpoints at any time N , for most real problems it is desirable to implement this model in more computationally efficient recombining two-dimensional lattices. This requires a few additional steps, but is also straightforward. Although the endpoints of the binomial nodes are recombining, when there are two separate factors, the procedure for capitalizing on the recurring values must be modified. To demonstrate how these values can be arranged in a two-dimensional lattice and to show the relationship to the values in a standard tree, the first two steps using, for example, the time zero equilibrium level ($\xi_0 = 2.857$, $\Delta\xi = 0.145$) and deviation ($\chi_0 = 0.119$, $\Delta\chi = 0.158$) from Schwartz and Smith (2000) are shown in Figure 3.

In the two-dimensional views in Figure 3, shifts in the equilibrium level are indicated by the column position, and shifts in the deviation from the equilibrium are indicated by the row position. In the standard branching view, the distribution of the endpoints and their duplicate values appears to be somewhat arbitrary; however, the two dimensional view lends intuition toward the development of an algorithm to build the lattice using two-dimensional arrays at each discrete time step.

An example of such an algorithm is presented by Clewlow and Strickland (2000) for the case of two GBM models and based on four branch nodes at each step. In this case, one of the GBMs has been replaced with a one-factor Ornstein–Uhlenbeck process, and the four-branch node has also been replaced with a two-node marginal-conditional sequence. However, it

Figure 3 Progression of Endpoints in Two Dimensions



is relatively straightforward to convert the two-node sequence back to a four-branch node after testing to see whether censoring is required or not.

4. Application to a Real Option Valuation Problem

In this section we apply this discrete model to example real option problems discussed by Schwartz and Smith (2000) in their paper introducing their two-factor model. These problems include a short-term investment and a long-term investment, both of which depend on the stochastic price of oil. In the short-term investment, a firm has an option to develop a project at a cost of \$40,000 which, if exercised, would deliver 1,000 barrels of oil per year on a 40% per year exponential decline. The long-term investment is also a development option, but of larger scale. This project has an exercise cost of \$800,000, an initial production rate of 5,000 barrels per year, and an exponential decline rate of 5%. There is also a three-year lag until commencement of production with this project, which is due to construction.

If the stochastic oil price model is calibrated to futures prices, it can be considered a risk-neutral forecast of future oil prices. When using a risk-neutral forecast for valuation, the future cash flows resulting from the project's options can be discounted at the risk-free rate (5%). For these examples, to facilitate comparison, we use the same risk-neutral forecast and two-factor model parameters that Schwartz and Smith (2000) use in their paper.

The value of these investments under optimal exercise for the case of an infinite-horizon option is obtained by Schwartz and Smith (2000) using the standard linear programming approach to the dynamic programming problem. In this section we solve a finite-horizon version of these problems, specifying that the option expires after nine years. This is more typical of real natural resources projects, where a firm obtains the development rights for a set period, after which the rights revert back to the mineral owner.

We solve these problems using methods from each of the three different general discrete time approaches: binomial lattices, trinomial lattices, and Monte Carlo simulation. For the binomial lattice approach we use

our two-factor binomial model. Because of the limitations of the trinomial lattice methods for multi-factor models based on Hull and White (1994) that were discussed earlier, and because they have not been applied to cases where one or more of the factors follow a GBM model, we sought to use a more general technique. Unfortunately, there is no trinomial lattice method outlined in the literature that utilizes standard branching probabilities, such as those derived from mean-variance matching, that can guarantee valid probabilities for a mean-reverting process. However, a trinomial lattice is essentially a special case of explicit finite difference dynamic programming. Therefore, we used a grid-based dynamic programming approach that is modified from the one Schwartz and Smith (2000) used for solving the infinite-horizon version of the problems in their paper. For Monte Carlo simulation, we implement the approach outlined by Longstaff and Schwartz (2001), which uses regression to approximate the value function. We then compare results from the three approaches and discuss development and computational requirements.

The dynamic programming method for solving the infinite-horizon version of the problems requires a grid, or two-dimensional state space for long-term equilibrium price and current deviation, along with transition probabilities between points in the grid between discrete time steps. Schwartz and Smith (2000) discretized the state space into a 35 by 35 grid (1,225 elements total) for these example problems and calculated the transition probabilities per a multivariate normal distribution from each point on the grid. Transitions were allowed to all possible states in the next step, and the boundaries were handled by assuming that transitions to values beyond the boundary in the untruncated model instead transition to the boundary in the truncated model. The linear formulation of the dynamic program has as many decision variables as states and twice that number of constraints, because in each state there are two actions: develop or wait.¹

The same state space and transition probabilities can be used in the finite version of the problem; however, there is no steady-state Markov decision process in that case. Instead, the problem is solved by

¹ The details of this algorithm were provided by Smith (2007).

starting in the last period, considering the optimal choice between exercising the option to develop and letting it expire in all possible states, and then rolling back the model to the initial time period, making the develop/wait decision at each time step. Computationally, this process involves the multiplication of transition matrices by value functions along with maximization to select the optimal course of action in each state.

We also obtained solutions for each problem using the simulation-regression method of Longstaff and Schwartz (2001). This method requires no discretization of the state space for a grid or lattice; rather, paths of the two-factor process for oil price are simulated, and at each step the develop/wait decision is made by comparing the value from exercising immediately versus the value for waiting that is predicted by regression on ex post payoffs from continuation. Over a large number of iterations, the value function for continuing will be well approximated, and the average value of the project from the simulation is the value of the project with its options.

The results from each of the three methods along with the values for immediate exercise are shown in Figure 4 for the short-term investment, and in Figure 5 for the long-term investment. Following the convention of Schwartz and Smith (2000), we present these results for a range of possible values for the two factors (equilibrium level and deviation). In each of these figures, the lower surface is the value from immediate exercise. The results with the option to wait from the three different methods are very

Figure 4 Value Function for Short-Term Investment

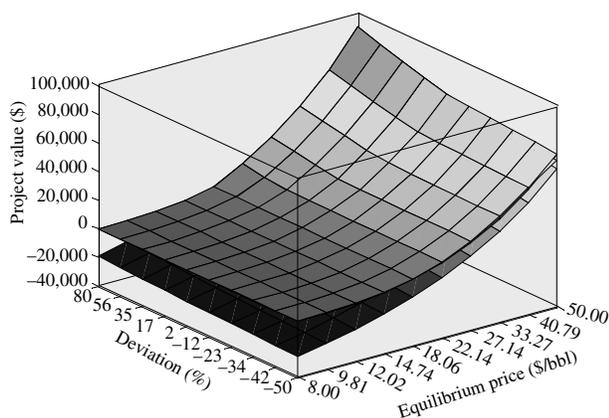
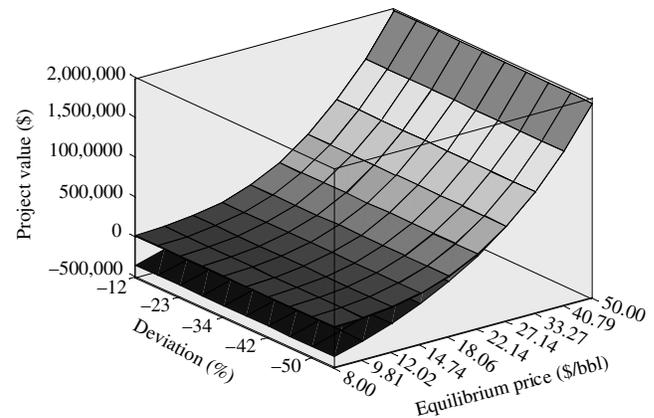


Figure 5 Value Function for Long-Term Investment



closely grouped together, showing general agreement between the three methods for valuing finite-lived options.

Across all 100 combinations for input equilibrium price and deviation, the grid-based dynamic programming approach results had a root mean squared error of \$923 for the short-term investment and \$5,038 for the long-term investment relative to the results from the simulation method. For perspective, these values correspond to 2.3% and 0.6% of the respective project investments. By comparison, the binomial method had root mean squared errors of \$710 and \$5,663 for the two respective investments, representing 1.8% and 0.7% of the investment levels.

Although these figures and statistics indicate general agreement among the three different methods and accuracy that is within the typical requirements for real options problems, we can further compare these methods in terms of convergence, computational time, and requirements for algorithm development and implementation. Each of the methods requires significant computational time or development effort in a particular area. The simulation-regression method is very flexible in terms of stochastic process modeling and does not require a scheme for discretizing the state space; however, it does require an approximation of the value function. The grid-based dynamic programming method requires a discretized state space and transition probabilities between all possible states, including boundaries, and achieves convergence as the state space and time increments become small. The binomial method

also requires a state space, albeit with fewer transition probabilities and no boundaries, and achieves convergence as time increments become small.

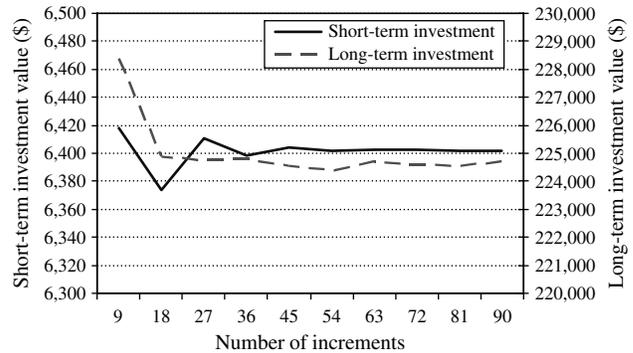
The relative computational disadvantage of simulation methods has been discussed in the literature by Tseng and Lin (2007), among others, and is due to the requirements of generating thousands of paths of the stochastic process, running regressions at each time step using data from these paths to determine the value function of continuing, and optimizing the decision making across all paths and at each step. For example, the simulation-regression method required nearly an hour to run the large number of iterations needed to produce a stable value for each equilibrium price–deviation combination in Figures 4 and 5. Therefore, we will focus our discussion of convergence and running time on the binomial and grid-based dynamic programming approaches.

For comparison, we selected starting values near the midpoint of the ranges shown in Figures 4 and 5 ($\xi_0 = 2.857, \chi_0 = 0.119$) and observed convergence and running times for both methods. Each of the methods tested here was implemented in Microsoft Excel with VBA (Visual Basic for Applications) coded algorithms. Implementation in other platforms (e.g., MatLab, C++) may differentially benefit the computational time for the different methods; however, our work is intended to represent a common ground comparison of both computational efficiency and development requirements. The implementations were tested on a standard PC with a 2.16 GHz processor and 1 GB of RAM.

The convergence behavior and computational times for the binomial method are shown in Figure 6 and Table 1, respectively. This figure and table display the typical oscillating convergence pattern for binomial models, with values for the lowest number of time increments within 1–2% of the stabilized value, and indicate that convergence was quickly obtained for these examples within just a few increment refinements. Running times for these increment levels were within one to two seconds for both examples.

For comparison, the convergence behavior and computational times for the grid-based dynamic programming approach are shown in Figure 7 and Table 2, respectively. In Figure 7, curves are shown for the highest level of granularity, which was 36 time

Figure 6 Convergence Behavior for Two-Dimensional Binomial Model



increments for both investments, along with the corresponding value from the binomial model.

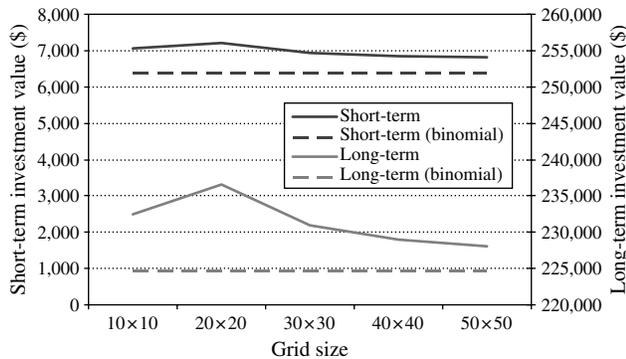
This figure and table show similar levels of convergence, although as shown in Table 2, this occurs across two model controls, time increments and grid size. We can observe from Figure 7 that convergence is more gradual for the dynamic programming model, and Table 2 indicates that grid size is the primary determinant of computational time. This table also shows longer running times relative to the binomial model, especially for the larger grid sizes, with running times of several minutes for the 50×50 grid model. This is because of the sheer size of the transition probability matrix, which in that case has dimensions of $2,500 \times 2,500$ and contains 6.25 million probabilities.

By default, the grid-based dynamic programming model provides not only the value for the selected starting points, $\xi_0 = 2.857$ and $\chi_0 = 0.119$, but also

Table 1 Computational Times for Two-Dimensional Binomial Model

Time increments	Lattice endpoints	Short-term investment		Long-term investment	
		Value (\$)	Running time (sec.)	Value (\$)	Running time (sec.)
9	100	6,148	0.3	228,374	0.4
18	361	6,374	0.6	224,884	0.8
27	784	6,411	0.9	224,749	1.4
36	1,369	6,398	1.6	224,795	2.7
45	2,116	6,404	2.9	224,562	5.1
54	3,025	6,402	4.5	224,386	8.7
63	4,096	6,402	6.9	224,696	13
72	5,329	6,402	10	224,583	20
81	6,724	6,402	14	224,558	28
90	8,281	6,402	20	224,698	38

Figure 7 Convergence Behavior for Grid-Based Dynamic Programming Model



for all other points on the grid. This also contributes to running times that are longer than the binomial method, which starts from a single point. This may be desirable for some applications; however, in others it may be preferable to have the option provided by the binomial model of starting from one point or many.

The amount and complexity of code required for implementation was also different for the three methods. We again focus on the grid-based dynamic

programming and binomial lattice methods, because these two approaches were observed to have a significant advantage in computational efficiency relative to the simulation-regression method. The most difficult tasks in applying the grid-based dynamic programming method are calculation of the transition probability matrices and operations with these matrices, and the value function to find the value function at each step in working backward through time. The transition probability matrices first require the calculation of the conditional distribution parameters (mean and standard deviation of joint normal distribution in the case of the Schwartz and Smith (2000) process) in the next step from each point on the state space grid. The transition probability matrix will then have dimensions $M \times N$, where M is the number of possible equilibrium price states, and N is the number of possible deviation states.

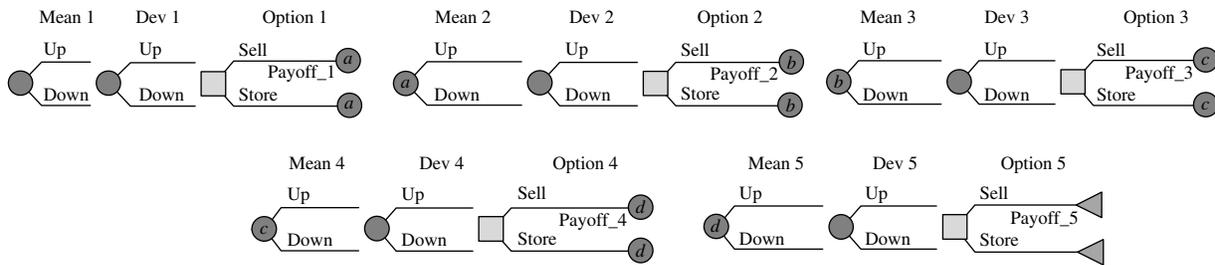
The key task in coding the binomial method is constructing the two-dimensional binomial lattice and calculating the probabilities of up and down movements of the two factors at each step. As discussed earlier, and as shown by Clewlow and Strickland (2000) for the case of two correlated GBM processes, this is a relatively straightforward extension of the Cox et al. (1979) binomial model which can be implemented in very concise (50 lines or less) coding.

Last, we note an important property of the two-dimensional binomial lattice method, which is its amenability to implementation in decision tree format using the marginal-conditional node sequence discussed earlier. This can be an important advantage for more complex applications, because decision trees can accommodate virtually any type and combination of options, including path-dependent options, simply by adding decision nodes and terminal payoff statements at the appropriate locations in the tree, as long as the underlying uncertainty or uncertainties are appropriately modeled. In the case of multiple underlying uncertainties, those uncertainties related to market risks should be specified in a risk-neutral manner using market information, whereas uncertainties related to private risks should be separately modeled and specified using appropriate objective estimates of the uncertainty (Smith and Nau 1995).

As an example, we consider a problem, again depending on the uncertain price of crude oil, for

Table 2 Computational Times for Grid-Based Dynamic Programming Model

Time increments	Grid size	Short-term investment		Long-term investment	
		Value (\$)	Running time (sec.)	Value (\$)	Running time (sec.)
9	10 × 10	7,634	4.4	246,364	9.3
	20 × 20	7,284	14	239,275	20
	30 × 30	7,207	37	237,744	44
	40 × 40	7,176	101	237,153	114
	50 × 50	7,160	208	236,875	235
18	10 × 10	7,805	5.8	248,376	11
	20 × 20	7,247	16	236,632	28
	30 × 30	7,109	48	233,752	65
	40 × 40	7,057	116	232,680	150
	50 × 50	7,031	235	232,179	285
27	10 × 10	7,569	6.4	242,964	13
	20 × 20	7,210	21	236,038	35
	30 × 30	7,008	60	231,800	85
	40 × 40	6,933	146	230,248	191
	50 × 50	6,897	264	229,532	345
36	10 × 10	7,050	6.6	232,460	15
	20 × 20	7,210	26	236,519	45
	30 × 30	6,945	71	231,001	103
	40 × 40	6,847	158	228,992	228
	50 × 50	6,801	300	228,061	410

Figure 8 Decision Tree Model of the Path-Dependent Option

which we use the same two-factor forecast as in the long-term and short-term investments. In this example, we seek to determine the value of storage options afforded a vertically integrated (production, storage, transportation, marketing, and refining) oil firm. In this case, the value of the option at each point in time and state space depends both on the price at that point and the path taken to that point, because the volume available to be sold depends on the exercise policy of options at all earlier points. This type of path-dependent option can easily be modeled with a decision tree, as illustrated in Figure 8 using the DPL™ software.

This model depicts a five-year project horizon, during which the price is modeled in each period as the evolution of the long-term equilibrium level and deviation, where the producer has the option to sell the amount of oil produced during that period or carry it forward, either financially or physically, to the subsequent periods based on expectations for future price levels and resulting payoffs. Unlike a recombining grid or lattice, this problem can be correctly modeled in a decision tree, because all possible paths to each endpoint are explicitly modeled. Assuming an initial production level of 10,000 barrels in the first period, with a 10% exponential decline thereafter and a \$15 per barrel production cost, and then solving the decision tree yields a value of \$185,860. This compares to a project value of \$137,286 with the immediate sale of all oil produced in each period, and this yields an option value of \$48,574 for the optimal timing of oil sales. Such nonrecombining trees are limited to about 30 total steps for practical computational times. Nonetheless, this example illustrates how this class of problems can now be solved under the assumption of an underlying two-factor stochastic process model.

5. Summary and Conclusions

We have shown how two-factor mean-reverting stochastic processes for underlying assets can be modeled in discrete time as two-factor binomial lattices. This method can be implemented with a relatively simple algorithm that leverages the primary advantage of recombining lattices; the number of states grows linearly with the number of periods. We have implemented this approach, as well as a simulation-regression method and a grid-based dynamic programming method, for two example finite-lived real option problems and obtained results with each method that are analogous to those Schwartz and Smith (2000) obtained for infinite-lived versions of these problems. Moreover, we have demonstrated that this approach is computationally efficient, especially compared to the simulation-regression approach for these problems. We also note that the two-dimensional binomial lattice algorithm is very concise and straightforward to develop, and the method also lends itself to the valuation of more complex options via modeling in a decision tree format.

Future research in this area might involve improving the computational performance of this approach and comparing it to a wider range of approaches, including those mentioned in the introduction. For example, although the binomial model demonstrated a high convergence rate in our tests, recent research (Chung and Shih 2007) suggests there is potential to obtain even more rapid convergence through the use of computational modifications to the standard binomial approach. As another example, Maller et al. (2006) discuss a recombining multinomial tree-based approach for the evaluation of American options on arbitrary Lévy processes that may offer an alternative for constructing a lattice to approximate a mean-

reverting stochastic process and therefore may be interesting to explore.

Appendix. Proof of Convergence for Two-Factor Diffusion

For convergence of the approximation to the general SDE

$$dY_t = \mu(Y, t) dt + \sigma(Y, t) dz,$$

the conditions are as follows:

(1) The functions $\mu(Y, t)$ and $\sigma(Y, t)$ are continuous and $\sigma(Y, t)$ is nonnegative.

(2) A solution of $Y_t = Y_0 + \int_0^t \mu(Y_s, s) ds + \int_0^t \sigma(Y_s, s) dz_s$ exists on $0 < t < \infty$ (this and condition (1) ensure that the limiting SDE is well behaved). When this condition is satisfied, the process $\{Y_t\}_{0 \leq t < T}$ is characterized by (a) the starting point Y_0 , (b) the continuity of Y_t , (c) the drift $\mu(Y, s)$, and (d) the diffusion $\sigma^2(Y, s)$.

Given this characterization, convergence of the discrete process $\{Y_t\}$ is proved by showing the following:

- (1) $Y_0 \rightarrow Y_0 \forall h$,
- (2) jump sizes of Y_0 become small at a sufficiently rapid rate,
- (3) $\mu_h(Y, s) \rightarrow \mu(Y, s)$, and
- (4) $\sigma_h^2(Y, s) \rightarrow \sigma^2(Y, s)$.

The paper by Stroock and Varadhan (1979) contains a detailed discussion of convergence requirements for discrete diffusion processes.

In this case, we have decomposed the two-factor process into an arithmetic Brownian motion (ABM) process for ξ and a conditional arithmetic Ornstein–Uhlenbeck process for χ , and we need to show convergence for both approximations. The proof of convergence for the binomial approximation of an ABM is well known and will therefore not be discussed in detail here.

Proof of convergence for the conditional Ornstein–Uhlenbeck process follows that shown in the appendix of Hahn and Dyer (2008), with the modification that the drift for the conditional process in the case of the Schwartz and Smith (2000) model is $\kappa(0 - \chi_t) + \rho_{\xi\chi}\sigma_\chi$, because the short-term deviation reverts to zero. Factoring in the possible need to censor the transition probabilities, the drift of the process is therefore

$$\mu_h(\chi, t) \equiv \begin{cases} \kappa\chi_t + \rho\sigma & \text{if } 0 < p_h < 1, \\ \sigma/\sqrt{h} & \text{if } p_h = 1, \\ -\sigma/\sqrt{h} & \text{if } p_h = 0. \end{cases}$$

With this modification, following the proof in Hahn and Dyer (2009), it is straightforward to verify that we have convergence to the instantaneous drift

$$\lim_{h \rightarrow 0} \mu_h(\chi, t) = \kappa\chi - \rho\sigma_\chi = \mu(\chi, t),$$

and we also likewise satisfy the variance condition

$$\lim_{h \rightarrow 0} \sigma_h^2(\chi, t) = \sigma^2(\chi, t).$$

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