RISK THEORY SEMINAR

Optimal Portfolio Selection when Constrained by Investment in a Mandatory Social Security Asset

ABSTRACT

The impact of a mandatory, government sponsored social security program is examined from the perspective of the individual's investment decision process within a continuous time portfolio framework. The inter-generational risk sharing characteristic of a public pay-as-you-go pension (modeled as a long-term stochastic process as opposed to the short-term processes associated with private investments) provides additional spanning opportunities and can be used to create a hedge portfolio for inter-temporally risk averse individuals. The approach results in a generalization of the Arrow-Pratt measure of risk aversion across assets by positing the existence of a similar measure for risk aversion across time. By contrasting the usual assumption of social security myopia, it is shown that the use of a dynamic portfolio selection process, wherein the individuals adjusts his overall portfolio to take account of the characteristics of the social security asset. Even when the investor views the mandatory level of investment in the social security asset as sub-optimal, he or she is able to minimize this effect by taking such characteristics into account via a dynamic portfolio selection process.

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I. Introduction

The impact of a mandatory, government imposed social security program on individual savings has received considerable attention over the years.[1] Economists, in examining the social security/private savings decision, have generally focused on both the two-period utility maximization problem, based on the labor leisure trade-off, or a life cycle model. Unfortunately, the models of consumer savings and portfolio selection utilized by the standard life cycle model have tended to be rather simplistic. They tend to view the social security asset and the decision to undertake an investment in the social security asses as a government mandate isolated from the individual's overall portfolio decision. Often, the social security asset is viewed as a poor investment compared to private investments. The expected return in lower than that of other assets which are assumed to dominate the social security asset. Moreover, individuals are viewed as being forced into a Pareto suboptimal state by investing a predetermined portion of income in the social security asset. In contrast, this article provides a more complete analysis, incorporating the value of an inter-generational hedge and allowing the individual to adjust his or her investment portfolio to account for the specific characteristics of a social security asset.

[1] See Thompson (1983) for a review of some of the key issues of social security. Abel (1986, 1987) and Gordon and Varian (1988) both model the inter-generational aspects of social security. More recently, Blocker, Kolokoff and Ross (2008) have reviewed the measurement of the cost of social security and Baumgart, Perun and Mensah (2008) the political issues associated with looking at the complementary roles of social security and private savings.

The inter-generational benefits of a social security asset cannot be replicated by private markets. Hedging against a stochastic shock to private investment returns may be optimal, if priced correctly, but both generations would have to be alive before the occurrence of a stochastic shock to create such a market (and the enforceability would be questionable). However, an impartial planner could surmise that as long as individuals across generations were risk averse, they would opt to insure against stochastic events, the timing of which cannot be accurately forecast. Altruistic, overlapping generations will choose to set up some kind of a social security system which provides an *a priori* benefit to all generations. There is a strong incentive for an initial generation, having suffered a shock, to implement such a program; if properly structured, future generations will not have the incentive to break the social contract. While some generations would *ex post* suffer a loss *vis-à-vis* a prior generations, they would not chose to revoke the contract as they could still potentially benefit from the risk pooling with future generations.[2]

This paper develops a more sophisticated model of individual portfolio selection based on some of the unique characteristics of the social security asset. The provision of an inter-temporal hedge within the investor's lifetime, quasi-inflation indexed annuities, with comparatively low transactions costs and the potential for an inter-generational hedge differentiate the social security asset from private annuities. Noting that the return on the social security asset is associated with the long-term growth in the economy, this asset is able to span a set of economic time-states which may not be available in private markets. This augmented spanning capability is modeled through the addition of a long-term stochastic process to the usual short term process which governs the returns on private investment. Some of the special characteristics of social

[2] In addition, the fact the median voter is older that the median taxpayer has often been used to explain the remarkable political stability of social security programs around the world.

security are of particular importance in determining optimal portfolio choice for the inter-temporally risk averse investor.

The social security asset provides investment opportunities which differ from those of a standard private annuity. In addition to smoothing consumption over time, the social security asset provides an annuity that is indexed to some measure of the standard of living. Under the assumption of continuous-time portfolio selection and recognizing the intergenerational pooling made possible though collective social security, this article demonstrates that a social security asset would provide an important investment hedge for individuals who are inter-temporally risk averse. Within a portfolio model, social security is simply viewed as one of a number of assets that the investor selects for his or her portfolio, albeit one with unique characteristics. The optimal level of all risky assets must be determined simultaneously subject to the additional constraint of a prescribed investment in the mandatory social security asset. The individual investor's optimizing behavior will treat the mandatory investment in social security as part of a hedge against inter-temporal risk and will adjust his holdings in other assets accordingly. It is also shown that the resulting portfolio can be separated into four quasi-mutual funds consisting of the risk free asset, the market portfolio, and two inter-temporal hedge portfolios: one a hedge of the short-term stochastic process associated with private investment returns; the other, a long-term stochastic process associated with overall economic growth.

Since participation in the social security system is mandatory for virtually all citizens and the benefit level (i.e., the rate of return on the social security asset) is set by fiat, it is likely that for many investors this level will be suboptimal (and at first glance significantly so). Nevertheless, while the particular level of investment in a government sponsored program will undoubtedly lead to distortions in labor markets and other investments and some loss in Pareto efficiency for the economy as a whole, the existence of fairly complete financial markets will allow many investors to offset a portion of these distortions.[3]

The remainder of this paper will proceed as follows. Section II will discuss some special characteristics of the social security asset. Section III will present the underlying assumptions of the model, while the formal model will be derived in section IV. Section V will then conclude with some observations about the ability of investors to change the composition of other assets in their investment portfolio to partially offset constraints imposed on the investment decision by the social security asset.

II. Characteristics of Social Security as an Investment

For the purposes of this paper, social security will refer exclusively to the system of periodic payments made by individual investors over their working lifetimes to purchase government provided annuities at the time of retirement. It is assumed that (a) contributions are actuarially fair; (b) investors are knowledgeable about the level of contributions and benefits; and (c) the social security system is in balance with regards to demographic changes. For simplicity, the redistributive tax associated with some social security systems (transfers from high to low income individuals) are considered as part of general taxation and not as part of the investment decision. Administration and other transactions costs are assumed to be small (of or below the order of other comparable investment programs) and can therefore safely be ignored.

[3] The argument is quite similar to the "homemade leverage" reasoning associated with the capital structure irrelevance propositions of Miller-Modigliani. With complete financial markets individuals can take short or long positions to back out the desired portfolio. Although the social security asset has some characteristics that cannot be fully replicated in private markets, the individual investor should be able to approximate his or her preferred position.

While the social security asset must be considered as one of many investment assets which are held in the investor's portfolio there are some specific characteristics which distinguish social security from other assets available to investors. First, and most important, investment in the social security asset is required for virtually all employees. Second, the level of participation and benefit levels are set by law. Third, the benefits of the program are contingent upon living to retirement age. They cannot be assigned or sold directly to third parties.[4] Fourth, since sponsored by the government, the return to the social security asset has no default risk. The value of the asset is assumed to fluctuate with the long term rate of economic growth. This differs from the "riskless" rate of return paid on government securities. While all government bonds are instantaneously risk free, these are more risky than the social security asset as they are fully subject to inflation and interest rate risk over time. Because of the implicit indexing, the social security asset provides an ideal instrument for hedging some types of inter-temporal risk. Fifth, the social security asset provides essential diversification to lower income individuals who may have most of their wealth concentrated in human (labor) capital which deteriorates rapidly once a certain age is reached. Finally, because of the implicit social contract associated with a government sponsored social security system, transfers can be made across generations with the effect of smoothing consumption across long-term economic cycles. Generations experiencing worse than average economic prosperity would be subsidized by more prosperous generations and *vice versa*. Since the shocks affecting the long-term economy are assumed to be stochastic in nature, risk averse individuals would be willing, *a priori*, to accept such an intergenerational risk pooling. These characteristics of social security are taken into account by the individual in his or her personal investment decision, and should likewise be considered by the government of in setting policy regarding the structure of Social Security cost and benefit levels.

[4] Even assuming that some individual would be willing to grant an assignment of the social security asset the bankruptcy laws would likely prevent enforcement. Moral hazard would preclude acceptance.

III. Assumptions of the Model

In order to keep the concentration of the model on the effect of Social Security on savings behavior a number of simplifying assumptions are made. First, in addition to the assumptions regarding the social security asset in the previous section, income and asset returns are considered net of taxes. Second, it is assumed that markets are perfect (i.e. no transactions costs, information regarding the distribution of asset returns are known and individuals are in agreement regarding this information set). Investment assets are also characterized by constant returns to scale, that is, the return to holding a single share of an asset is proportional to holding N shares of the same asset. Third, it is assumed that markets are complete, trading takes place continuously, and that individuals make investments in order to maximize their Von Neuman-Morgenstern utility functions.

IV. Derivation: Continuous Time Portfolio Choice Model

The portfolio selection problem is typically modeled as the maximization of expected utility from consumption across the investor's lifetime plus the utility of the individual's final bequest. In continuous time the maximization problem is written as follows[5]:

$$MAX\ [C,X]\quad E_0\left[\int_0^T U(C;t,y,l)dt + B(W;T,y,l)dt\right] \qquad [1]$$

where:

E_0 = The Expectation operation taken at time t = 0
U = Utility of instantaneous consumption flow
B = Utility of final bequest at time T
C = Instantaneous consumption flow
W = Wealth

[5] See Merton (11), (12), and (13) and Ingersoll (9) for a more complete development of the continuous time portfolio selection models.

X = Vector of investment portfolio weights ($x_{1,\ldots,}x_N$)
y = Index of short term state of economy
1 = Index of long term state of economy
t = Index of time
T = Terminal period of consumption

Both the consumption and bequest utility functions are assumed to be well behaved (i.e. $U_C > 0$; $U_{CC} < 0$; $B_C > 0; B_{CC} < 0$). To avoid the problem of negative consumption and to have the utility bounded above, we assume that zero consumption would have infinite negative utility $(U(0,t) = -\infty)$ and that the marginal utility of additional consumption when consumption is infinitely high adds no additional utility$(U'(\infty,t) = 0)$. The investor forms his or her portfolio from N risky securities, an instantaneously risk free asset and the social security asset. Free choice is allowed in selecting the optimal combination of the risk free asset and the risky assets, but the level of investment in the social security asset is fixed collectively at x_S. With x_i is invested in each risky asset and x_s in the social security asset, all remaining wealth $\left(1-\sum x_i - x_s\right)$ will be invested in the risk free asset. It is important to note that the investment in each risky asset and in the risk free asset can be positive, zero, or negative depending on whether the investor holds the asset or holds a short position in the asset (borrowing is equivalent to holding a short position in the risk free asset), however investment in the social security asset is fixed.

At this point, it is assumed that the states of the economy are fully characterized by two independent stochastic processes, one short-term and the other long-term. The return on the risk free asset and the return on each risky investment assets are based on the short term stochastic process of the economy while the return on the social security asset is based on the long term stochastic process. The two processes underlying the economy are assumed to be general wiener processes, with instantaneous drift and diffusions terms. These processes are fully described by the following partial differential equations:

$$\textit{Short term}: \quad dy = \mu_y(y,t)dt + \sigma_y(y,t)dz_y \quad [2]$$

$$\textit{Long term}: \quad dl = \mu_l(l,t)dt + \sigma_l(l,t)dz_l \quad [3]$$

In each equation the dt term represents the instantaneous drift of the process per unit of time, while the dz term refers to the instantaneous variation or diffusion of the process (the dt terms are of order dz^2). Knowledge of the properties of these two stochastic equations will allow us to develop an equation for the processes underlying the rates of return on the riskfree asset, each risky asset and the social security asset.

The stochastic process underlying the instantaneous rate of return on the riskfree asset "R" is a function of the short term state of the economy and time. This can be written as:

$$\frac{dR}{R} = r_f(y,t)dt \quad [4]$$

The stochastic process governing the rate of return on the risky assets is also based on the short term state of the economy and time. This process will be written in terms on the change in price "P_i" of each risky asset:

$$\frac{dP_i}{P_i} = \alpha_i(y,t)dt + \sigma_i(y,t)dz_i \quad [5]$$

The stochastic process behind the rate of return on social security wealth "S" provides a riskless rate of real consumption and therefore depends on the long term expected state of the economy and time. The special characteristics of the social security asset discussed earlier (especially the ability to smooth the return across economic cycles and transfer wealth across generations) justifies the use of a separate underlying process. The social security process can therefore be written as:

$$\frac{dS_i}{S_i} = r_s(l,t)dt + \sigma_s(l,t)dz_s \qquad [6]$$

These three stochastic processes will govern the growth in individual wealth and the riskiness of the investors underlying portfolio. Together with the requirement that all wealth is invested, $\left(\sum_i x_i + x_s + x_f = 1\right)$, they can be used to form the continuous time budget constraint for the maximization process. We state formally that the maximization problem is subject to the following constraints:

$$dW = \left\{\left[\sum_{i=1}^{N} x_i(\alpha_i - r_f) + x_s(r_s - r_f) + r_f\right]\cdot W - C\right\}dt + \left[\sum_{i=1}^{N} x_i\sigma_i dz_i + x_s\sigma_s dz_s\right]\cdot W \qquad [7]$$

$$C \geq 0 \qquad [8]$$

where:

dW	= Instantaneous change in total wealth
W	= Total Wealth[6]
C	= Instantaneous rate of consumption
r_f	= Instantaneous riskless rate of interest
x_i	= Proportion of portfolio invested in risky security i
α_i	= Instantaneous rate of return on asset i
σ_i	= Instantaneous diffusion of risky asset i
x_s	= Proportion of portfolio invested in the social security asset
r_s	= Instantaneous rate of return on social security
σ_s	= Instantaneous diffusion of social security asset
N	= Number of risky securities

[6] Since we are looking at wealth as a stock rather than as a flow, wealth includes the net present value (risk adjusted) of future earnings. One of the risky assets will therefore be the investment in human capital.

The principal constraint [equation 7], states that the change in wealth over an infinitesimally short period of time is equal to the return on the investment portfolio (including the risk free asset rate, the risky assets and the social security asset), less consumption during the same time period. The second constraint [equation 8], is a requirement that consumption be positive at all times.[7]

The solution to the maximization problem is found using the following method. Assuming that all future decisions are made optimally a Bellman function is established:

$$J(W;y,l,t) = \underset{C,T}{\text{MAX}}\, E_t \int_t^T U(C;y,l,t)\,dt + B(W_T,T) \qquad [9]$$

This is identical to the utility maximization problem presented in equation [1] except that the expectation is taken at an arbitrary future time t. The Bellman function can be thought of as the derived utility from investment. The same constraints, equations [7] and [8] will also apply to the Bellman equation. Since we are assuming that the investor will always make the optimal decision in the future, the investor can concentrate on the immediate investment/consumption decision. The original maximization problem can be rewritten as:

$$\begin{aligned} &\underset{C,T}{\text{MAX}}\, E_0 \int_0^{dt} U(C;y,l,t)\,dt + \int_{dt}^T U(C;y,l,t)\,dt \\ &= \underset{C,T}{\text{MAX}}\, E_0 \{U(C;y,l,t) + J(W+dW;y,l,t+dt)\} \end{aligned} \qquad [10]$$

[7] Usually the choice of utility function and market constraints will take care of the second constraint. Otherwise, the second set of constraints will have to be verified at each step of the solution. The existence of the social security asset will generally insure that consumption will always be positive since this asset pays off in those states where earnings abilities have diminished.

To solve the maximization problem, we expand the Bellman function taking the conditional expectation. Applying a multidimensional version of Ito's lemma to equation [9] results in:

$$J(W;y,l,t) = \underset{C,X}{\text{MAX}}\, E_0 \left\{ \begin{array}{c} U(C;y,l,t) + J(W;y,l,t) + J_W dW \\ + \frac{1}{2} J_{WW} (dW)^2 + J_y dy + \frac{1}{2} J_{yy} (dy)^2 \\ + J_l dl + \frac{1}{2} J_{ll} (dl)^2 + J_t dt + J_{Wy} dW dy \\ + J_{Wl} dW dl + J_{yl} dy dl \end{array} \right\} \qquad [11]$$

Replacing dW, dy and dl with the stochastic processes given in equations [4], [5], and [6]; then taking the expected value and subtracting J(W;y,l,t) from each side gives the following form of the standard Hamilton Jacobi Equation:

$$\begin{aligned} \underset{C,X}{\text{MAX}}\, U(C;y,l,t) + J_W \left\{ \left[\sum_i x_i \left(\alpha_i - r_f \right) + x_s \left(r_s - r_f \right) + r_f \right] \cdot W - C \right\} dt \\ + \tfrac{1}{2} J_{WW} \left[\sum_i \sum_j x_i x_j \sigma_{ij} + x_s^2 \sigma_s^2 + \sum_i x_i x_s \sigma_i \sigma_{is} \right] \cdot W^2 dt \\ + J_y \mu_y dt + \tfrac{1}{2} J_{yy} \sigma_y^2 dt + J_l \mu_l dt + \tfrac{1}{2} J_{ll} \sigma_l^2 dt + J_t dt + J_{yl} \sigma_{yl} dt \\ + \left\{ J_{Wy} \left[\sum_i x_i \sigma_{iy} + x_s \sigma_s \sigma_y \right] + J_{Wl} \left[\sum_i x_i \sigma_{il} + x_s \sigma_s \sigma_l \right] \right\} \cdot W dt = 0 \end{aligned} \qquad [12]$$

Taking the derivatives of this partial differential equation with respect to consumption and with respect to the portfolio weight of each investment asset we can obtain an analytic solution for the optimal portfolio. Two cases of the solution will be derived. The first, “Investment Under Social Security Myopia” assumes that the investor ignores the effect of social security as a hedge portfolio. This is the assumption which underlies much of the economic analysis of savings decisions when participation in social security is mandated. The solution is identical to the straightforward, textbook application of continuous time portfolio selection. The resulting portfolio is precisely what would be expected in the case of the investment decision without a social security asset except that only a portion $W_I = [1 - x_s]W$ of the investors wealth is optimized. The portfolio separation which results matches the separation which is obtained under the assumption of a single short term stochastic process. Since the social security is ignored, the resulting portfolio will be suboptimal for virtually all investors. The second case, “Social Security as a Hedge Portfolio,” develops a more realistic portfolio selection decision where the relationship between the social security asset and the risky assets is explicitly taken into account by the decision maker. The portfolio separation which results takes account of the additional hedging opportunities afforded by the existence of a social security asset.

Case I: Investment Under Social Security Myopia

The investor who is myopic regarding the effect of the social security asset in his portfolio decision essentially disregards the impact of the long terra stochastic process underlying the economy. For this investor the Hamilton Jacobi Equation would be restated as follows (where W_I represents investable wealth):

$$\begin{aligned}\underset{C,X}{\text{MAX}}\, U(C; y, t) + J_W \left\{\left[\sum_i x_i\left(\alpha_i - r_f\right) + r_f\right]\cdot W - C\right\} dt \\ + \tfrac{1}{2} J_{WW}\left[\sum_i \sum_j x_i x_j \sigma_{ij}\right]\cdot W^2 dt + J_y \mu_y dt + \tfrac{1}{2} J_{yy}\sigma_y^2 dt \\ + J_t dt + J_{yl}\sigma_{yl} dt + J_{Wy}\left[\sum_i x_i \sigma_{iy} dt\right]\cdot W dt = 0\end{aligned} \quad [13]$$

The derivatives of the Hamilton Jacobi equation with respect to consumption and with respect to each risky asset are given by:

$$/C: \quad U_C - J_w = 0 \quad [14]$$

$$/x_i: \quad J_w\left(\alpha_i - r_f\right) + J_{WW}\sum_j x_j \sigma_{ij} W_I + J_{Wy}\sigma_i\sigma_y = 0 \quad [15]$$

The initial first order condition equation restates the standard Fisherian investment consumption decision — the marginal utility of current consumption is equated to the utility derived from investment (i.e., the expected marginal utility of future consumption). Equation [15] can be solved to obtain the investment weights. The second term of each equation is a linear function of the x_i's weighted by the covariance of the stochastic processes of each pair of risky assets. It is therefore possible to solve for the optimal portfolio weights. Let $\sum$ be the covariance matrix of these stochastic processes:

$$\sum = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & & \ddots & \vdots \\ \sigma_{n1} & \cdots & \cdots & \sigma_{nn} \end{bmatrix} \qquad [16]$$

We can then define v_{ij} as the corresponding element of the inverse of the covariance matrix.

$$\boldsymbol{V} = \sum^{-\mathbf{1}} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & & \ddots & \vdots \\ v_{n1} & \cdots & \cdots & v_{nn} \end{bmatrix} \qquad [17]$$

Solving for the portfolio weights we find:

$$x_i = -\frac{1}{W_I}\frac{J_w}{J_{ww}}\sum_i v_{ij}\left[\alpha_i - r_f\right] - \frac{1}{W_I}\frac{J_{w_y}}{J_{ww}}\sum_j v_{ij}\,\sigma_{ij} \qquad [18]$$

Substituting the initial first order condition we can interpret the factors containing investable wealth and the derivatives of the derived utility function as measures of risk aversion.

$$-\frac{1}{W_I}\frac{J_w}{J_{ww}} = -\frac{1}{W_I}\frac{U_C}{U_{CC}} \qquad \left[\begin{array}{l}\text{Standard Arrow-Pratt measure} \\ \text{of Risk Aversion applied to} \\ \text{Diversification across assets}\end{array}\right. \qquad [19]$$

$$-\frac{1}{W_I}\frac{J_{W_y}}{J_{ww}} = -\frac{1}{W_I}\frac{U_C}{U_{Cy}} \qquad \left[\begin{array}{l}\text{A similar measure of} \\ \text{Risk Aversion applied to} \\ \text{Diversification across time}\end{array}\right. \qquad [20]$$

The result of the social security myopia case is a separation of the investor's portfolio into three separate funds. The first portfolio will be an investment in the risk free rate. The second will be the market portfolio (identical to the tangency portfolio in the Sharpe-Litner two period mean-variance capital asset pricing model). The final portfolio is the hedge portfolio, the portfolio of assets that will possess the maximum correlation with the short term state variable. This portfolio will be used to hedge against inter-temporal risk by providing a high return when investment opportunities are poor and a low return when investment opportunities are good[8]. The remainder of the individual's investable funds is invested at the risk free rate. The amount invested in the market portfolio and the risk free asset will be determined by the investor's risk aversion across assets (relative to that of the market) while the amount of his portfolio invested in the hedge portfolio will depend on his or her degree of inter-temporal risk aversion.

For the myopic investor, the result of the investment decision is invariant with respect to the existence of the social security asset. Any contributions which the investor makes to social

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[8] If, for example, interest rates were based on the short-term stochastic process, a long-term bond could be considered for inclusion in the hedge portfolio. When interest rates fall the price of the bond can be expected to rise and vice versa.

security, reduces his investable wealth during his or her working life but asset weights do not adjust for the hedge opportunity provided. Under the assumption of myopic social security investment, it is apparent that mandatory investment in the social security asset will result in a suboptimal portfolio for most, if not all, investors. The investor, in effect, determines both consumption and investment decisions independent of the level of the social security benefit. Thus, to the extent that the social security asset provides an inter-temporal hedge which is not taken account of, the inter-temporally risk averse investor's investment in the hedging portfolio will be higher than optimal and investment in the market portfolio will be lower than optimal. The less inter-temporally risk averse investor will face the opposite result. The assumption of social security myopia is manifestly irrational. The investor who does not take account of the returns on the social security asset and particularly the correlation with the state variable in making his investment decision is not maximizing total utility. We need a model which takes account of the total portfolio decision.

Case II: Social Security as an Inter-temporal Hedge

The investor who recognizes that the social security asset has both an impact on wealth and an important role as a hedge portfolio will maximize the full Hamilton Jacobi partial differential equation. His investment in each asset will depend on the level of mandatory investment in the social security asset (x_s). Recall that the investment decision was given in equation 12 as:

$$\begin{aligned}\underset{C,X}{\text{MAX}}\; U(C;y,l,t) + \; & J_W \left\{\left[\sum_i x_i\left(\alpha_i - r_f\right) + x_s\left(r_s - r_f\right) + r_f\right]\cdot W - C\right\} dt \\ & +\tfrac{1}{2} J_{WW}\left[\sum_i \sum_j x_i x_j \sigma_{ij} + x_s^2\,\sigma_s^2 + \sum_i x_i x_s \sigma_i\,\sigma_{is}\right]\cdot W^2 dt \\ & + J_y \mu_y\, dt + \tfrac{1}{2} J_{yy}\,\sigma_y^2 dt + J_l \mu_l dt + \tfrac{1}{2} J_{ll} \sigma_l^2 dt + J_t\, dt + J_{yl}\,\sigma_{yl}\, dt \\ & + \left\{J_{Wy}\left[\sum_i x_i\,\sigma_{iy} + x_s\sigma_s\sigma_y\right] + J_{Wl}\left[\sum_i x_i\,\sigma_{il} + x_s\sigma_s\sigma_l\right]\right\}\cdot W dt = 0\end{aligned} \qquad [21]$$

Taking the derivatives with respect of consumption and portfolio weights results in the following first order conditions:

$$/C: \quad U_C - J_w = 0 \qquad [22]$$

$$/x_i: \quad J_w\left(\alpha_i - r_f\right) + J_{WW}\sum_j \left[x_j\,\sigma_{ij} + x_s\sigma_s\right]\cdot W \; + J_{Wy}\,\sigma_{iy} + J_{Wl}\sigma_{il} = 0 \qquad [23]$$

Equation [22] differs from the myopic solution in that total wealth (not just investable wealth) is taken into account. The marginal utility from present consumption is still equated to marginal utility arising from investment (future consumption). The effect of taking account of the social security asset in the portfolio decision is more pronounced when we consider the N first order conditions given in equation [23]. Each equation contains two covariance terms in addition to the terms which comprise the first order conditions of the myopic solution. First, there is a term consisting of the covariance between each risky security and the social security asset weighted by wealth and the second derivative of derived utility with respect to wealth; and second, there is a term which embodies the covariance between each investment asset and the long term stochastic process of the economy weighted by the cross derivative of the derived utility function with respect to wealth and the long term state variable. Since the J_{WW} term is still a linear function of the x_i's, as before, we can solve the system of equations by inverting the covariance matrix $\sum$ to obtain the v_{jj} terms "as in equations [16] and [17]. Solving for the portfolio weights we find:

$$x_i = -\frac{1}{W}\frac{J_w}{J_{ww}}\sum_i v_{ij}\left[\alpha_i - r_f\right] - \frac{1}{W}\frac{J_{w_y}}{J_{ww}}\sum_j v_{ij}\,\sigma_{ij}$$
$$-\frac{1}{W}\frac{J_{w_{yl}}}{J_{ww}}\sum_l v_{ij}\,\sigma_{il} - \sum_l v_{ij}\,x_s\sigma_{is} \qquad [24]$$

As before, x_s is invested in the social security asset, and $1 - \sum x_i - x_s$ is invested in the risk-free rate. Comparing the portfolio weights in this equation to the "social security myopia" results (equation [18]), we note two important differences. First, we now have separation into four portfolios. The existence of the social security asset which is based on a different (long term) stochastic process has resulted on the creation of an additional hedge portfolio. The investor who is inter-temporally risk averse will now separate his portfolio into four quasi-mutual fund portfolios. The first will be the risk free asset, the second the market portfolio, the third the portfolio which has the maximum correlation with the short term state variable and the fourth the portfolio with the maximum correlation with the long term state variable. This fourth portfolio will contain the social security asset (possibly along with any other assets that are able to hedge inter-temporal risk between the short and long term stochastic processes).

Additionally, the investor will decrease his investment in each risky asset by an amount equal to $x_s \sum_i \sigma_{is} v_{ij}$. This represent the percent of the investor's portfolio which is invested in the social security asset weighted by the relationship between the covariance of the return of the asset with the social security portfolio and the covariance of the return of the risky asset with each other risky asset. This can be thought of as the investment beta of each risky asset with the social security asset; a higher $\boldsymbol{\beta_s}$ will result in lower levels of investment in the risky asset while a lower $\boldsymbol{\beta_s}$ will have the effect of higher investment in the risky assets. The net amount that the investor will place in each portfolio will depend on his or her risk aversion with respect to each type of risk as well as the level of wealth of the investor[9]. If we assume decreasing absolute risk aversion, we would expect that the wealthier would have a lower relative demand for hedging instruments including the social security asset, while the poor, who are also less able to diversify their wealth, will have a higher demand for hedging portfolios and especially social security.[10]

V. Conclusions and Observations

This article is a first attempt to model the investment demand for social security as part of the overall investment objectives of the individual investor. The model as presented, by taking account of some special hedging characters of the social security asset, may help explain investment behavior under the constraint of mandatory social security. In particular, the model predicts that the social security asset will provide inter-temporally risk averse investors with an additional hedge portfolio which could not be reproduced in private capital markets. The model also indicates how the investor will take account of social security wealth when forming his or her utility maximizing investment portfolio. This is more realistic that the typical "dead weight loss" analysis assigned to a mandatory social security investment. When the investor receives additional utility from the structure of asset payoffs, or is able to adjust his or her entire portfolio, a significant portion of any non-optimality may be eliminated at a relatively low cost.

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[9] If the social security asset is the principal asset in the long term hedging portfolio, the investment may also be constrained by the limits placed on the investment in this asset.

[10] This relative valuation helps explain the persistent support for social security by all but those that are most likely to not need the inter-temporal hedge. It also points out the relative ease which those that are less risk averse or have a lower need for hedging can offset the negative impact of a mandatory public pension investment.

The assumptions of the model are of necessity extreme. The rate of return on the social security asset will likely be affected by long term economic growth as well as short term budgetary requirements, but it is not considered unrealistic to assume that, as a first approximation, real social security benefits will follow a much smoother path than the risk adjusted return on other investments. Considering that a large segment of the population has a significant percentage of wealth concentrated in the social security asset and the continued widespread political support for social security lends credence to the hypothesis that individuals value the hedging capability of the social security asset.

The analysis provided can be particularly valuable to government planners in emerging markets that are looking at the establishment of modification of government pension programs. A pure private model, as proposed by many capital market proponents in emerging economies, may be suboptimal for a majority of the population when compared to a multi-pillar approach when inter-temporal hedges are ignored.

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