Liquidity and Expected Returns in a Multi-Factor Asset Pricing Model

Preliminary Version

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Abstract

In this paper I show how the price impact of trading affects the cross-section of expected returns. I analyze a market with a large number of assets, risk avers investors and asymmetric information about firm specific components of future dividends. If there are no frictions then price impact is irrelevant for expected returns. If some investors are not able to short-sell then the factors that determine the dividends of illiquid assets carry a higher risk premium than comparable factors that determine the dividends of liquid assets. Liquidity is not a source of systematic risk beyond the systematic dividend risk factors.

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I Introduction

In this paper I analyze how liquidity affects the cross-section of expected returns. I define liquidity as the degree to which an individual investor impacts the price of an asset when he trades. The model in this paper ignores the bid-ask spread.

A key insight of standard asset pricing models is that compensation for risk does not depend on the total risk of an asset but instead on the components of the return that are correlated with the marginal utility of consumption. This result has been demonstrated for example within the framework of the CAPM by Sharpe (1964) and Lintner (1965) or for more general cases by Rubinstein (1974) and Lucas (1978). These models derive pricing kernel equations from the first order conditions of portfolio optimization problems. Since these first order conditions take prices as given these models ignore the potential effects of price impact on expected returns.

In this paper I incorporate the price impact of trading into a multi-factor asset pricing model. I develop a model where investors trade a large number of firms whose dividends depend on systematic and idiosyncratic components. This model produces the following main results:

1. The dividend factors determine the cross-section of expected returns. If there are no frictions then liquidity is irrelevant for expected returns.

2. If some investors are not able to short-sell then the factors that determine the dividends of illiquid assets carry a higher risk premium than comparable factors that determine the dividends of liquid assets.

Price impact has no direct effect on expected returns in this model for the following reason: trading an individual asset is costly because of the price impact, but investing the same dollar amount in a large diversified portfolio is cheap since the price impact on large portfolios is small.[1] The price of a systematic dividend risk factor is given by the price of a diversified portfolio that replicates this factor. Since the price impact of trading this replicating portfolio is small, liquidity does not directly affect the premium for systematic dividend risk. Therefore, in a large frictionless market where idiosyncratic risk can be diversified away, liquidity has not direct effect on expected returns.

Even though liquidity does not directly affect expected returns it is possible that short sale constraints generate an indirect effect of liquidity on returns. To understand this indirect effect assume investors receive private information about the idiosyncratic components of dividends. Noise traders prevent prices from revealing all information. This information asymmetry does not directly affect expected returns since investors can diversify the idiosyncratic dividend risk in large portfolios. However, the information asymmetry affects the cost of trading *individual* assets: assets with low information asymmetry (liquid assets) can be traded with low price impact. Illiquid firms have high information asymmetry and therefore high price impact.

Since liquid firms have a lower price impact than illiquid firms investors trade liquid firms more aggressively. Suppose an investor receives a positive private signal about the idiosyncratic dividend component of a particular firm. If the investor overweighs this firm in his portfolio he also automatically overweighs the systematic factors that determine

[1]Subrahmanyam (1991) also argues that trading a basket of securities is cheaper than trading individual securities.

the dividend of this firm. Since the investor trades liquid assets more aggressively he will overweigh the systematic dividend factors of a liquid firm more aggressively than the dividend factors of an illiquid firm. The demand for individual liquid assets therefore also creates a demand for the systematic risk factors that determine the dividends of these assets.

If there are no frictions in the market then an investor who overweighs a particular firm can adjust his factor exposure by short selling a diversified portfolio that loads on the same risk factors. This short-selling counterbalances the effect of the demand for individual assets on these risk factors. Therefore the demand for liquid firms does not affect the premium that investors require for systematic dividend risk factors. Hence in a market without frictions liquidity is irrelevant for expected returns.

If there are some investors in the economy who cannot short-sell then these investors overweigh the dividend risk factors that are associated with liquid assets. Market clearing therefore requires that other investors underweigh these risk factors. In the equilibrium factors that determine the dividends of illiquid firms therefore have a higher risk premium than comparable factors that determine the dividends of liquid assets. In this way the demand for liquid assets indirectly affects expected returns if at least some investors are not able to short-sell.

A key feature of this paper is that that the large number of risky assets allows investors to diversify idiosyncratic risk. Several other authors have also examined the effects of liquidity on expected returns in models with multiple risky assets. In Holmstrom and

Tirole (2001) risk-neutral corporations demand liquid assets to satisfy future cash needs. Eisfeldt (2004) solves a model where risk-avers investors sell claims to private production technologies. The market for these asset is illiquid due to asymmetric information. One of the differences between Eisfeldt and my model is that there is no aggregate risk in Eisfeldt whereas my model focuses on the pricing of aggregate risk. In Vayanos (2004) and Aharya and Pedersen (2005) risk-avers investors face exogenous transaction costs.[2] Weil (2008) solves a model where risk-neutral investors need time to search for trading partners.[3] In Brunnermeier and Pedersen (2008) risk avers investors who experience individual liquidity shocks trade multiple risky assets with a group of risk neutral investors. The liquidity shocks impact prices because the risk avers investors arrive at the market sequentially and because the risk neutral investors face wealth constraints so that they cannot perfectly accommodate the liquidity shocks.[4]

Admati (1985) solves a rational expectations equilibrium model with multiple risky assets and asymmetric information. Easley and O'Hara (2004) and Hughes, Liu and Liu (2005) examine the effect of information asymmetry on expected returns in markets with multiple assets.

Several authors have analyzed how liquidity affects the expected return in a market with a single risky asset. For example Garleanu and Pedersen (2004) examine the expected return of a single risky asset with endogenous transaction costs. Baker and Stein (2004)

[2]Other models of expected returns with exogenous transaction costs include Constantinides (1986), Amihud and Mendelson (1986), Heaton and Lucas (1996), Vayanos (1998), Vayanos and Vila (1999), Jacoby, Fowler and Gottesman (2000), Huang (2003), and Lo, Mamaysky and Wang (2004).

[3]Other search-based models that examine expected returns include Vayanos and Wang (2007) and Duffie, Garleanu and Pedersen (2007).

[4]Other models with a single risky asset and risk-avers investors who do not all trade simultaneously include Grossman and Miller (1988) and Huang and Wang (2008a, 2008b)

argue in a model with a single risky asset that high liquidity indicates that there are irrational investors present and that the asset is overvalued.

The remainder of this paper is organized as follows. Section II describes the setup of the model. Section III derives an equilibrium where investors trade firms with lower information asymmetry more aggressively than firm with higher information asymmetry. Section IV analyzes how short-sale constraints can generate a higher risk premium for illiquid firms than for liquid firms. Section V concludes.

II Setup of the Model

A. Overview

There are two time periods, $t = 0$ and $t = 1$. The economy consists of an infinite number of firms and an infinite number of investors. At time $t = 1$ each firm j pays a random dividend D_j. At time $t = 0$ investors can trade the firms at equilibrium prices P_j. The dividends D_j depend on common factors f and on idiosyncratic components $z_j = z_{1j} + z_{2j}$. There are two groups of firms, group A and group B. Before trading starts investors publicly observe the idiosyncratic dividend components z_{1j} for firms in group A, but not for firms in group B. In addition to the public information each investor i also observes a noisy private signal $\hat{z}_{i2j}$ about the dividend component z_{2j} of one particular firm j. Some investors act as noise traders. The private signal of the noise traders has no informative value. The noise traders erroneously believe that their signals are meaningful. Investors are randomly matched with firms such that each firm is associated with exactly one private signal $\hat{z}_{i2j}$. After investors receive their private information they submit trade schedules to an auctioneer. The auctioneer then determines a market clearing equilibrium prices P_j. When investors submit their trade schedules they take the fact into account that their own demands might impact the prices at which these demands are executed.

B. Number of assets and market clearing

In order to understand the infinite setting consider first a finite economy.[5] Assume there are N firms and $k \times N$ investors. Before trading starts at $t = 0$ each investor owns one share of each firm. Hence each firm has kN shares outstanding. There is also a single risk-free asset which is in zero net supply. The price of the risk-free asset will be arbitrary in this model. To simplify the notation I set the risk-free rate to zero. Each firm j pays at time $t = 1$ a total dividend of D_j. Hence the dividend per share is given by $\frac{D_j}{kN}$. The total price or market capitalization of firm j at time $t = 0$ is given by P_j. Hence the price per share is given by $\frac{P_j}{kN}$.

Suppose that an investor holds a portfolio containing half of the shares outstanding of firm $j = 1$ and containing one share of each of the remaining firms. Then the cost of this portfolio is given by

$$\text{cost } = \frac{0.5kN}{kN}P_1 + \sum_{j=2}^{N} \frac{1}{kN}P_j = 0.5P_1 + \sum_{j=2}^{N} \frac{1}{kN}P_j.$$

Accordingly the payoff of the portfolio is given by

$$\text{payoff } = 0.5D_1 + \sum_{j=2}^{N} \frac{1}{kN}D_j.$$

[5] For another model with a countable infinite set of assets see Chamberlain and Rothschild (1983).

In a large market with infinitely many firms these expressions become

$$\begin{aligned} \text{cost} &= 0.5P_1 + \lim_{N\to\infty} \sum_{j=2}^{N} \frac{P_j}{kN} \\ \text{payoff} &= 0.5D_1 + \lim_{N\to\infty} \sum_{j=2}^{N} \frac{D_j}{kN} \end{aligned}$$

(assume for now that these series converge to some random variable with probability one). This example demonstrates that in a large market we need to distinguish between buying a large stake in a particular firm and buying a small stake in a large number of firms. In order to distinguish between these two cases I will identify an arbitrary portfolio of risky assets with a pair of sequences

$$\Big\{\mathbf{I}, \mathbf{X}\Big\} = \Big\{\{I_j\}_{j=1}^{\infty}, \{X_j\}_{j=1}^{\infty}\Big\},$$

where $X_j \in \mathbf{R}$ and $I_j \in \{0, 1\}$. The variable I_j indicates the meaning of X_j: if $I_j = 1$ then X_j refers to a percentage of shares outstanding, if $I_j = 0$ then X_j refers to a particular number of shares. For example in the case above we have $\mathbf{I} = \{1, 0, 0, \ldots\}$ and $\mathbf{X} = \{0.5, 1, 1, \ldots\}$. With this notation the general cost and payoff of a portfolio are given by

$$\begin{aligned} \text{cost} &= \lim_{N\to\infty} \sum_{j=1}^{N} \left(I_j + \frac{1 - I_j}{kN}\right) X_j P_j \\ \text{payoff} &= \lim_{N\to\infty} \sum_{j=1}^{N} \left(I_j + \frac{1 - I_j}{kN}\right) X_j D_j \end{aligned}$$

with probability one. Since investors start with an initial endowment of one share per firm their final wealth is given by

$$W_1\Big(\{\mathbf{I},\mathbf{X}\},\mathbf{P}\Big) = \lim_{N\to\infty}\sum_{j=1}^{\infty}\frac{D_j}{kN} + \lim_{N\to\infty}\sum_{j=1}^{N}\left[\left(I_j + \frac{1-I_j}{kN}\right)X_j(D_j - P_j)\right] \qquad (1)$$

where $\mathbf{P} = \{P_j\}_{j=1}^{\infty}$ and $\{\mathbf{I},\mathbf{X}\}$ is the net trade. Let $\big\{\mathbf{I}_i,\mathbf{X}_i\big\} = \big\{\{I_{ij}\}_{j=1}^{\infty},\{X_{ij}\}_{j=1}^{\infty}\big\}$ be the trade of investor i. For a fixed N the market clears for firm j if the aggregate net trade equals zero:

$$\sum_{i=1}^{kN}\Big(I_{ij}kN + (1-I_{ij})\Big)X_{ij} = 0.$$

Hence the market clearing condition for asset j in a large market is given by

$$\lim_{N\to\infty}\sum_{i=1}^{kN}\left(I_{ij} + \frac{1-I_{ij}}{kN}\right)X_{ij} = 0 \qquad (2)$$

with probability one. Equations (1) and (2) show that the individual assets in a diversified portfolio do not affect the final wealth or the market clearing condition. An investment in an individual asset only has an effect if the investor buys a significant portion of the shares outstanding. This property will imply that investors act as price takers when they buy a large portfolio of assets, but consider their price impact when they make a large investment in a particular asset.

For simplicity I will set $k = 1$ in the following, so that there is exactly one investor per firm.

C. Dividends

The firms are divided into two groups with identical population weights. One half of firms is in group A and the other half is in group B. Let I_{jA} be an indicator variable that equals one if $j \in A$ and zero otherwise. Let I_{jB} be an indicator variable that equals one if $j \in B$ and zero otherwise. The dividend of firm j is given by

$$D_j = \alpha + I_{jA} f_A + I_{jB} f_B + z_j \tag{3}$$

where α is a constant, f_A and f_B are factors and z_j is an idiosyncratic component. The idiosyncratic component is given by

$$z_j = z_{1j} + z_{2j}. \tag{4}$$

This decomposition will be important for the firm specific information that investors receive. For simplicity I assume that the factors f_A and f_B have two possible realizations, $-\sigma_f$ and $\sigma_f > 0$. These realizations occur with probability 1/2. Hence each factor has mean zero and variance

$$E[f_A^2] = E[f_B^2] = \sigma_f^2.$$

The probability that f_A and f_B have identical realizations is given by

$$\text{Prob}\big[f_A = f_B\big] = \frac{1+\rho}{2}, \qquad \rho \in [-1, 1].$$

Hence the correlation between the two factors is given by

$$\frac{\mathrm{E}[f_A f_B]}{\sqrt{\mathrm{E}[f_A^2]\mathrm{E}[f_B^2]}} = \rho. \tag{5}$$

The model therefore includes the case where a single factor determines all dividends ($\rho = 1$). The idiosyncratic components z_{1j} and z_{2j} are distributed accordingly: each z_{nj}, $n = 1, 2$ has the realization $-\sigma_z$ and $\sigma_z > 0$ with probability 1/2. Hence each z_{nj} has mean zero and standard deviation σ_z. All z_{nj} are pairwise independent and they are independent of f_A and f_B. To prevent dividends from becoming negative I assume that $\alpha > \sigma_f + 2\sigma_z$.

D. Information

There is more public information available about firms in group A than firms in group B. For each asset in group A all investors observe the realization of z_{1j}. For assets in group B the realization of z_{1j} is not publicly observable. I do not model the reason for this difference of public information. One possible motivation is that firms in group A produce products that are more interesting to consumers and therefore receive more media attention than firms in group B.

In addition to public information investors also observe private information about the idiosyncratic payoff components z_{1j} and z_{2j}. The private signal of investor i about firm j is given by

$$S_{ij} = \{\hat{z}_{i1j}, \hat{z}_{i2j}\} = \{z_{1j}, \hat{z}_{i2j}\} \tag{6}$$

where $\hat{z}_{inj} \in \{-\sigma_z, \sigma_z\}$ for $n = 1, 2$. The private signal $\hat{z}_{i1j}$ equals the realization of z_{1j} (which equals the public signal for $j \in A$) with probability one. The signal $\hat{z}_{i2j}$ is either informative or complete noise. An informative signal $\hat{z}_{i2j}$ coincides with the realization of z_{2j} with probability one. A noise signal coincides with probability one half. The signal is informative with probability q. Hence the probability that investor i's signal $\hat{z}_{i2j}$ equals the realization of z_{2j} is given by

$$\text{Prob}[\hat{z}_{i2j} = z_{2j}] = q + (1 - q)\frac{1}{2} \quad (7)$$

For simplicity I assume that each investor observes a private signal about one particular firm and that each firm is associated with exactly one informed investor. Hence there is one private signal given by (6) per firm and there are no firms without private signals.

Private information is only valuable for investors if they are able to trade based on this information. A standard way to enable information-based trading is to introduce some form of noise, as for example in Grossman and Stiglitz (1980). Since the purpose of this model is to explain which assets investors trade I would like to generate all trading decisions endogenously. An example of endogenous noise traders are the discretionary liquidity traders in Admati and Pfleiderer (1988). However, investors who experience liquidity shocks will optimally trade the market portfolio in this model. Liquidity or wealth shocks can therefore not generate firm-specific trading in a model with many risky assets.

To generate endogenous noise trading for individual assets I assume that there are two groups of investors: one half of all investors are rational and the other half are noise traders. The probability of a correct private signal $\hat{z}_{i2j}$ is given by $q = 0$ if investor i is a noise traders and $q \in (0,1]$ if i is a rational investors. The rational investors know that they are rational. The noise traders believe that they are rational with probability one. Noise traders are distributed randomly within the population of investors. Hence for any given firm j the signal $\hat{z}_{i2j}$ is a noise signal ($q = 0$) with probability one half. This information structure is common knowledge.

E. Definition of the Equilibrium

Let $\mathbf{R}^\infty$ be the set of all infinite sequences of real numbers. Let $\{0,1\}^\infty$ be the set of all infinite sequences who consist only of the elements 0 or 1. For any given private signal S_{ij} of investor i about firm j a trade schedule $\{\mathbf{I}_i, \mathbf{X}_i\} = \{\mathbf{I}_i(\mathbf{P}; S_{ij}), \mathbf{X}_i(\mathbf{P}; S_{ij})\}$ is a mapping $\mathbf{R}^\infty \to \{0,1\}^\infty \times \mathbf{R}^\infty$. This mapping associates with each possible sequence of prices $\mathbf{P} = \{P_j\}_{j=1}^\infty$ a sequence of indicator variables $\mathbf{I}_i = \{I_{ij}\}_{j=1}^\infty$ and a sequence of demands $\mathbf{X}_i = \{X_{ij}\}_{j=1}^\infty$.

Each investor i submits a trade schedule $\{\mathbf{I}_i(\mathbf{P}; S_{ij}), \mathbf{X}_i(\mathbf{P}; S_{ij})\}$ to the auctioneer. The auctioneer then chooses a sequence of prices $\mathbf{P}$ that satisfy the market clearing condition (2) for all assets j. To emphasize that these equilibrium prices depend on the trade schedules I will sometimes write $\{P_j(\{\mathbf{I}_i, \mathbf{X}_i\}_{i=1}^\infty)\}_{j=1}^\infty$ for this sequence of prices. To complete the description of the pricing function I also assume that if such a sequence does not exist or if more than one sequence exists then no trading occurs and all investors keep their initial endowments and all prices will be set to the unconditional expectations of the dividends $P_j = \mathrm{E}[D_j] = \alpha$ (this case will not occur in the equilibrium).

Investors maximize expected utility of their final wealth W_1 in (1). All investors have identical utility functions $U(x)$ with $U'(x) > 0$ and $U''(x) < 0$.

Definition 1 (Optimal trades)**.** *For a given collection of trade schedules* $\{\mathbf{I}_h, \mathbf{X}_h\}_{h=1}^{\infty}$ *a trade schedule* $\{\mathbf{I}_i, \mathbf{X}_i\}$ *is* optimal *for investor* i *if the limits in* (1) *exist and if for any trade schedule* $\{\mathbf{I}'_i, \mathbf{X}'_i\}$

$$E_i\left[U\Big(W_1\left(\{\mathbf{I}_i, \mathbf{X}_i\}, \{P_j(\{\mathbf{I}_h, \mathbf{X}_h\}_{h=1}^{\infty})\}_{j=1}^{\infty}\right)\Big|S_{ij}\Big)\right]$$
$$\geq\ E_i\left[U\Big(W_1\left(\{\mathbf{I}'_i, \mathbf{X}'_i\}, \{P_j(\{\mathbf{I}'_h, \mathbf{X}'_h\}_{h=1}^{\infty})\}_{j=1}^{\infty}\right)\Big|S_{ij}\Big)\right] \tag{8}$$

where $\{\mathbf{I}'_h, \mathbf{X}'_h\}_{h=1}^{\infty}$ *equals the sequence* $\{\mathbf{I}_h, \mathbf{X}_h\}_{h=1}^{\infty}$ *with the element* $\{\mathbf{I}_i, \mathbf{X}_i\}$ *replaced by* $\{\mathbf{I}'_i, \mathbf{X}'_i\}$*.*

Sometimes it will be helpful to analyze the demand of an investor who acts as a price taker and ignores his influence on the equilibrium prices.

Definition 2 (Optimal trades for price takers)**.** *For a given sequence of random variables* $\mathbf{P}$ *a trade schedule* $\{\mathbf{I}, \mathbf{X}\}$ *is optimal for a price taker if the limits in* (1) *exist and if for any trade schedule* $\{\mathbf{I}', \mathbf{X}'\}$

$$E_i\left[U\Big(W_1\left(\{\mathbf{I}, \mathbf{X}\}, \mathbf{P}\right)\Big|S_{ij}\Big)\right]\ \geq\ E_i\left[U\Big(W_1\left(\{\mathbf{I}', \mathbf{X}'\}, \mathbf{P}\right)\Big|S_{ij}\Big)\right] \tag{9}$$

In the equilibrium each trade schedule $\big\{\mathbf{I}_i(\mathbf{P}; S_{ij}), \mathbf{X}_i(\mathbf{P}; S_{ij})\big\}$ is optimal given the trade schedule of all other investors $h \neq i$.

Definition 3 (Equilibrium)**.** *An equilibrium is a collection of trade schedules* $\{\mathbf{I}_i, \mathbf{X}_i\}_{i=1}^{\infty}$ *such that for each investor* $i \in \{1, 2, \dots\}$ *the trade schedule* $\{\mathbf{I}_i, \mathbf{X}_i\}$ *is optimal in the sense of* (8)*.*

This equilibrium definition is similar in spirit to the equilibrium definition in Kyle (1989) on page 322 (for the case of one risky asset). In the equilibrium of Definition 3 investors are able to condition their demand for any given firm j not only on the price P_j of firm j but also on the prices of all other firms in the economy. It will turn out that this assumption is not necessary. In the equilibrium described in Section III investors will achieve their optimal portfolios by conditioning their demand (I_j, X_j) for for firm j only on the price of firm j.

III Equilibrium

A. Conditional Expectations of the Idiosyncratic Dividend Components

Each investor i receives a signal $\hat{z}_{i2j}$ about the idiosyncratic dividend component z_{2j} for firm $j = j_i$. The probability that investors receive the correct signal is given by (7) for rational investors. Each investor i believes that he is rational with probability one. Since the distribution of z_j and $\hat{z}_j$ are symmetric around zero the conditional probabilities for investor i are given by

$$\text{Prob}_i[z_{2j} = y | \hat{z}_{i2j} = y] = \text{Prob}_i[\hat{z}_{i2j} = y | z = y] = q1 + (1-q)\frac{1}{2} = \frac{1+q}{2}, \qquad (10)$$

where $y \in \{-\sigma_z, +\sigma_z\}$. Hence the expectation conditional on observing the high signal realization $\hat{z}_{i2j} = +\sigma_z$ is given by

$$\mathrm{E}_i\left[z_{2j}\middle|\hat{z}_{i2j} = \sigma_z\right] = \sigma_z\big(2\mathrm{Prob}_i[z_{2j} = \sigma_z|\hat{z}_{i2j} = \sigma_z] - 1\big) = q\sigma_z$$

for investor i. In general we have

$$\mathrm{E}_i\left[z_{2j}\middle|\hat{z}_{i2j}\right] = q\hat{z}_{i2j}. \tag{11}$$

Hence investor i weights his signal with the probability that he observes an informative signal. Suppose investor $h \neq i$ observes the signal $\hat{z}_{i2j}$ of investor i. Since investor h does not know whether i is a rational investor or a noise trader the conditional probabilities for investor h are given by

$$\mathrm{Prob}_{h\neq i}[z_{2j} = y|\hat{z}_{i2j} = y] = \frac{\mathrm{Prob}_i[z_{2j} = y|\hat{z}_{i2j} = y] + \frac{1}{2}}{2} = \frac{2+q}{4} \tag{12}$$

Hence the expectation of z_{2j} conditional on investor $h \neq i$ observing the signal of investor i is given

$$\mathrm{E}_{h\neq i}\left[z_{2j}\middle|\hat{z}_{i2j}\right] = \frac{1}{2}q\hat{z}_{i2j}. \tag{13}$$

Investor $h \neq i$ weights the signal of investor i only with $\frac{q}{2}$ since $h \neq i$ does not know whether i is rational.

B. Conjectured Equilibrium Prices

Since there is exactly one informed investor for each firm and since there is no other form of noise I will start the search for an equilibrium by conjecturing that the equilibrium reveals all private signals. Since the dividends have a factor structure I will conjecture that the equilibrium prices have a corresponding factor structure:

$$P_j = \alpha + I_{jA} P_A + I_{jB} P_B + P_{z_j} \tag{14}$$

where the constant α and the indicator variables I_{jA} and I_{jB} are defined in (3), and P_A and P_B are constants, and the idiosyncratic price component is given by

$$P_{z_j} = \mathrm{E}_{h\neq i}\big[z_j \big| z_{1j}, \hat{z}_{i2j}\big] = z_{1j} + \mathrm{E}_{h\neq i}\big[z_{2j} \big| \hat{z}_{i2j}\big] \tag{15}$$

where $z_j = z_{1j} + z_{2j}$ as defined in (4). Then the dollar returns for each asset j are given by

$$D_j - P_j = I_{jA}\big(f_A - P_A\big) + I_{jB}\big(f_B - P_B\big) + (z_j - P_{z_j}). \tag{16}$$

By (13) the idiosyncratic dollar return for firm j is given by

$$z_j - P_{z_j} = \begin{cases} \big(+1 - \tfrac{1}{2}q\big)\, \hat{z}_{i2j} & \text{for } z_{2j} = \hat{z}_{i2j} \\[2ex] \big(-1 - \tfrac{1}{2}q\big)\, \hat{z}_{i2j} & \text{for } z_{2j} \neq \hat{z}_{i2j} \end{cases} \tag{17}$$

The probabilities of (17) conditional on observing $\hat{z}_{i2j}$ are given by (10) for investor i and by (12) for investor $h \neq i$. Hence the conditional expectations of the idiosyncratic returns are given by

$$E_i\left[z_j - P_{z_j}\middle|\hat{z}_{i2j}\right] = \frac{1}{2}q\hat{z}_{i2j} \tag{18a}$$

$$E_{h\neq i}\left[z_j - P_{z_j}\middle|\hat{z}_{i2j}\right] = 0 \tag{18b}$$

C. Optimal Final Wealth

Since the signals $\hat{z}_{i2j}$ are independent across j, the idiosyncratic price components P_{z_j} are independent across assets. Hence these price components do not affect the value of large equally weighted portfolios. Suppose an investor buys X_A shares of each firm $j \in A$ and X_B shares of each firm $j \in B$. The market value of this portfolio is given by

$$\lim_{N\to\infty}\left(\sum_{j=1}^{\frac{N}{2}} X_A \frac{\alpha + P_A + P_{z_j}}{N} + \sum_{j=\frac{N}{2}+1}^{N} X_B \frac{\alpha + P_B + P_{z_j}}{N}\right) = X_A \frac{\alpha + P_A}{2} + X_B \frac{\alpha + P_B}{2} \tag{19}$$

with probability one. Similarly the investor can also diversify the dividend residuals z_j away. Hence the payoff of the portfolio is given by

$$\lim_{N\to\infty}\left(\sum_{j=1}^{\frac{N}{2}} X_A \frac{\alpha + f_A + z_j}{N} + \sum_{j=\frac{N}{2}+1}^{N} X_B \frac{\alpha + f_B + z_j}{N}\right) = X_A \frac{\alpha + f_A}{2} + X_B \frac{\alpha + f_B}{2}. \tag{20}$$

with probability one. Assume that investor i also buys a fraction X_{j_i} of all shares outstanding of asset j_i. Then his final wealth as defined in (1) is given by

$$W_1 = \begin{cases} X_0 + \left(\frac{X_A}{2} + X_{j_i}\right)\left(f_A - P_A\right) + \frac{X_B}{2}\left(f_B - P_B\right) + X_{j_i}\left(z_{j_i} - P_{z_{j_i}}\right) & \text{for } j_i \in A \\ X_0 + \left(\frac{X_B}{2} + X_{j_i}\right)\left(f_B - P_B\right) + \frac{X_A}{2}\left(f_A - P_A\right) + X_{j_i}\left(z_{j_i} - P_{z_{j_i}}\right) & \text{for } j_i \in B \end{cases} \tag{21}$$

with probability one, where

$$X_0 = \alpha + \frac{P_A + P_B}{2}$$

is the value of the initial endowment of one share per firm. Equation (21) shows that the demand X_{j_i} for asset $j_i \in A$ or $j_i \in B$ does not affect the exposure to the factor f_A or f_B since the investor can freely choose his factor exposure by adjusting X_A or X_B. For example, if the investor would like to buy a large amount of firm $j_i \in A$ then he might short sell the remaining assets in A. By (18a) the idiosyncratic returns $z_j - P_{z_j}$ have expectation zero for $j \neq j_i$ conditional on the information of investor i. Hence investors avoid exposure to these residuals since they only add noise to the portfolio (Rothschild and Stiglitz, 1970).

Lemma 1 (Optimal portfolios do not contain idiosyncratic noise)**.** *Suppose prices are given by* (14) *and* (15)*. Suppose investor i observes the signal $\hat{z}_{i2j}$ about firm $j = j_i$. If the investor acts as a price taker (Definition 2) then there exist $X_A, X_B, X_{j_i} \in \mathbf{R}$ such that the optimal final wealth is given by* (21)*.*

Since the factors f_A and f_B have zero expectation investors invest a positive amount in these factors only if the prices P_A and P_B are negative. Since the factors are identically distributed investors invest a large portion of their wealth into the factor with the lower price.

Lemma 2 (Relation between optimal portfolios and prices)**.** *Under the conditions of Lemma 1 we have*

$$Sign\big[P_A - P_B\big] = \begin{cases} Sign\big[X_B - (X_A - X_{j_i})\big] & \text{for } j_i \in A \\ Sign\big[(X_B - X_{j_i}) - X_A\big] & \text{for } j_i \in B \end{cases} \tag{22}$$

and

$$\text{if } P_A = P_B \text{ then } \begin{cases} X_B > 0 \Leftrightarrow P_B < 0 & \text{for } j_i \in A \\ X_A > 0 \Leftrightarrow P_A < 0 & \text{for } j_i \in B \end{cases} \tag{23}$$

For a proof see Appendix A.

D. Information-based Trading for Firms with Low Information Asymmetry in Group A

The idiosyncratic return $z_j - P_{z_j}$ in (17) is independent of the factors f_A and f_B. The expected idiosyncratic return for investor i who observes $\hat{z}_{i2j}$ is given by (18b). Therefore investor i will buy shares of firm j if $\hat{z}_{i2j} > 0$. Since the realizations of $z_j - P_{z_j}$ for the case $\hat{z}_{i2j} > 0$ equal the realizations of $-(z_j - P_{z_j})$ for the case $\hat{z}_{i2j} < 0$ investor i will sell an equivalent amount of shares if $\hat{z}_{i2j} < 0$. Hence the demand of investor i for asset j_i is given by

$$X_{j_i} = \begin{cases} +X_Z & \text{if } \hat{z}_{i2j} > 0 \\ \\ -X_Z & \text{if } \hat{z}_{i2j} < 0 \end{cases} \tag{24}$$

where $X_Z > 0$ is a constant. Hence for each firm j in group A there is exactly one investor i who either buys or sells a fraction of X_Z of the shares outstanding. If investor i buys a fraction of X_Z shares then the remaining investors $h \neq i$ have to sell this fraction in aggregate. Since each investor has an initial endowment of one share, investors sell a fraction of X_Z shares outstanding in aggregate if each individual investor $h \neq i$ sells X_Z shares. If the equilibrium prices are given by (14) then investors can implement these demands by buying

$$X_j(P_j) = \begin{cases} +X_Z & \text{if } P_j = \alpha + P_A + z_{1j} - \frac{1}{2}q\sigma \\ \\ -X_Z & \text{if } P_j = \alpha + P_A + z_{1j} + \frac{1}{2}q\sigma \\ \\ 0 & \text{otherwise} \end{cases} \tag{25}$$

shares of each firm $j \in A$. The product

$$X_j(P_j) \times (D_j - P_j) = X_j(P_j) \times (f_A - P_a + z_j - P_{z_j}). \tag{26}$$

has expectation zero (conditional on the information of $h \neq i$) and is independent across firms. Therefore the payoff from investing $X_j(P_j)$ in all assets $j \in A$ is given by

$$\lim_{N\to\infty} \sum_{j=1}^{\frac{N}{2}} \frac{X_j(P_j) \times (D_j - P_j)}{N} = 0$$

with probability one. Investors are therefore able to accommodate the individual demands $X_{j_i} = \pm X_Z$ in (24) without affecting their final wealth W_1 in (21). Hence the demands $X_j(P_j)$ do not affect the optimal choice for the factor demands X_A and X_B.

E. Information-based Trading for Firms with High Information Asymmetry in Group B

Investors observe publicly the realization of the dividend component z_{1j} for all firms $j \in A$, but not for firms $j \in B$. Suppose investor i observes the signal $S_i = \{z_{1j}, \hat{z}_{i2j}\}$ about firm $j \in B$. If the equilibrium reveals all signals then the idiosyncratic returns for all firms in A and B are given by (17). Therefore, if investor i acts as a price taker his demand for firm $j \in B$ is given by (24). But this demand depends only on the second component of his signal $\hat{z}_{i2j}$ and not on the first component z_{1j} since the price P_j fully adjusts to z_{1j}. But if the demand X_{j_i} does not depend on z_1 then z_1 cannot be incorporated into the equilibrium price P_{j_i}.

One possibility for the equilibrium price to incorporate the private signals might be that investor i conditions his demands on the price. For example if he observes $(z_{1j} =$

$+\sigma_z, \hat{z}_{i2j} = +\sigma_z\big)$ then he submits the demand schedule

$$X_{j_i}\big(z_{1j} = +\sigma_z, \hat{z}_{i2j} = +\sigma_z\big) = \begin{cases} +X_Z & \text{if } P_j = \alpha + P_A + \sigma_z + \frac{1}{2}q\sigma_z \\ \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

If he observes $\big(z_{1j} = -\sigma_z, \hat{z}_{i2j} = +\sigma_z\big)$ he submits

$$X_{j_i}\big(z_{1j} = -\sigma_z, \hat{z}_{i2j} = +\sigma_z\big) = \begin{cases} +X_Z & \text{if } P_j = \alpha + P_A - \sigma_z + \frac{1}{2}q\sigma_z \\ \\ 0 & \text{otherwise} \end{cases} \tag{28}$$

and accordingly he submits $-X_Z$ for $\hat{z}_{i2j} = -\sigma_z$. If investors $h \neq i$ submit demands with the opposite signs conditional on the four possible prices then the market clears. These demand schedules are optimal for investors $h \neq i$. However they are not optimal for investor i since i can increase his expected utility by submitting $X_{j_i}\big(z_{1j} = -\sigma_z, \hat{z}_{i2j} = +\sigma_z\big)$ even if he observed $z_{1j} = +\sigma_z$. Investors $h \neq i$ therefore need to provide an incentive for i to reveal his signal.

Assume investor i holds a portfolio $\{\mathbf{I}, \mathbf{X}\}$ which contains a fraction $X_{j_i} = x$ of firm $j = j_i \in B$. Let

$$E\left[U\big(x\big)\middle| z_{1j}, \hat{z}_{i2j}, P_j = p, \mathbf{P}\right] = E\left[U\left(W_1\Big(\big\{\mathbf{I}, \mathbf{X}\big\}, \mathbf{P}, X_j = x\Big)\right)\middle| z_{1j}, \hat{z}_{i2j}, P_j = p, \mathbf{P}\right]$$

be the expected utility of this investor conditional on observing z_{1j}, $\hat{z}_{i2j}$ and $\mathbf{P}$, where $P_j = p$ and $j = j_i \in B$.

Lemma 3 (Separating demands)**.** *Assume investor i observes the private signal $S_{ij} = \{z_{1j}, \hat{z}_{i2j}\}$ for $j = j_i \in B$. There exists an $x^* \in (0, X_Z)$ such that for all $x \in [0, x^*]$*

$$E_i\left[U\big(X_Z - x\big)\bigg| z_{1j} = \sigma_z, \hat{z}_{i2j} = \sigma_z, P_j = \alpha + P_A - \sigma_z + \frac{1}{2}q\sigma_z, \mathbf{P}\right]$$
$$\leq \quad E_i\left[U\big(X_Z\big)\bigg| z_{1j} = \sigma_z, \hat{z}_{i2j} = \sigma_z, P_j = \alpha + P_A + \sigma_z + \frac{1}{2}q\sigma_z, \mathbf{P}\right]$$

and

$$E_i\left[U\big(-(X_Z - x)\big)\bigg| z_{1j} = -\sigma_z, \hat{z}_{i2j} = -\sigma_z, P_j = \alpha + P_A + \sigma_z - \frac{1}{2}q\sigma_z, \mathbf{P}\right]$$
$$\leq \quad E_i\left[U\big(-X_Z\big)\bigg| z_{1j} = -\sigma_z, \hat{z}_{i2j} = -\sigma_z, P_j = \alpha + P_A - \sigma_z - \frac{1}{2}q\sigma_z, \mathbf{P}\right]$$

with strict equality only for $x = x^*$.

Suppose investor i observes $\hat{z}_{12j} = +\sigma_z$. Then the investor would like to buy X_Z shares, preferably at the low price in (28) corresponding to $z_{1j} = -\sigma_z$. Lemma 3 shows that investors $h \neq i$ can force investor i to reveal his signal by selling $X - x$ shares at the low price corresponding to $z_{1j} = -\sigma$ and selling X shares at the high price corresponding to $z_{1j} = +\sigma$:

$$X_j(P_j) = \begin{cases} \pm X_Z & \text{if } P_j = \alpha + P_A \mp (\sigma_z + \frac{1}{2}q\sigma_z) \\ \\ \pm(X_Z - x^*) & \text{if } P_j = \alpha + P_A \mp (-\sigma_z + \frac{1}{2}q\sigma_z) \end{cases} \tag{29}$$

and $X_j(P_j) = 0$ otherwise, where the sign of the demand corresponds to the opposite sign inside the price equation. Given these demand functions the optimal demand of investor i is the market order

$$X_{j_i} = \begin{cases} \pm X_Z & \text{if } \hat{z}_{i2j} = \pm\sigma_z \text{ and } z_{1j}\hat{z}_{i2j} > 0 \\ \\ \pm(X_Z - x^*) & \text{if } \hat{z}_{i2j} = \pm\sigma_z \text{ and } z_{1j}\hat{z}_{i2j} < 0 \end{cases} \tag{30}$$

where the sign of the demand corresponds to the sign of $\hat{z}_{i2j}$. If x^* is given by Lemma 3 then investor i is indifferent between the demand in (30) and in (24) and we can therefore assume that he chooses (30).

F. How Investors Adjust their Factor Exposure

Equations (25) and (29) show how investor i accommodates the demands X_{j_h} given by (24) and (30) of other investors $h \neq i$. In addition investor i also needs to adjust his demand X_A or X_B to his own demand X_{j_i} in order to achieve his optimal exposure to the factors f_A and f_B. As shown in (21) investor i can eliminate the systematic risk from his investment in firm j_i by buying $-X_{j_i}$ shares of all other firms $j \in A$ or $j \in B$, $j \neq j_i$. Suppose all investors adjust their holdings accordingly. These adjustments have expectation zero and are independent across investors by (24) and (30). Hence the adjustments do not affect market clearing since for each firm $j \in \{A, B\}$

$$\lim_{N\to\infty} \sum_{i=1}^{N} (1 - I_{ij}) X_{ij_i} = 0$$

with probability one.

G. Adding it all up: the Total Demand of Investor i

The market clearing condition (2) implies that investors rationally act as price takers when they trade large diversified portfolios. Since the demands for the factors f_A and f_B are completely independent of the trades that are induced by private information $\hat{z}_{i2j}$, equation (22) in Lemma 2 implies that the market only clears if the factor prices P_A and P_B are identical.

Lemma 4 (Equilibrium demands)**.** *Suppose prices are given by* (14)*.*

1. *If the market clears then $P_A = P_B < 0$ and $X_A = X_B$ for all investors.*

2. *Suppose $X_A + X_{j_i} = X_B = 1$ is optimal for all investors who observe $\hat{z}_{i2j}$, $j \in A$ and $X_A = X_B + X_{j_i} = 1$ is optimal for all investors who observe $\hat{z}_{i2j}$, $j \in B$. Then the trade schedule $\left\{\mathbf{I}_i(\mathbf{P}; S_{ij}), \mathbf{X}_i(\mathbf{P}; S_{ij})\right\}$ where*

   (a) $I_{ij} = 1$ for $j = j_i$ and $I_{ij} = 0$ otherwise,

   (b) for $j \in A, j = j_i$: $X_{ij} = X_{j_i}$ given by (24)*,*

   (c) for $j \in A, j \neq j_i$: $X_{ij} = -X_{j_i} + X_j(P_j)$ given by (24) *and* (25)*,*

   (d) for $j \in B, j = j_i$: $X_{ij} = X_{j_i}$ given by (30)*,*

   (e) for $j \in B, j \neq j_i$: $X_{ij} = -X_{j_i} + X_j(P_j)$ given by (30) *and* (29)*,*

   generates the optimal final wealth in (21)*.*

3. *If all investors submit the trade schedules of part 2 then the market clears.*

Hence investors submit market orders when they speculate on their private signal $\hat{z}_{i2j}$ and condition on prices when they buy their diversified portfolios. Since all investor choose equal exposure to the factors f_A and f_B, the equilibrium factor exposures must be given by their initial endowments of one share per firm. It remains to be shown that there exist factor prices P_A and P_B such that holding the initial endowments is indeed optimal.

H. Equilibrium Factor Prices

Plugging the demands of Lemma 4 into the wealth equation (21) we get the equilibrium final wealth

$$W_1 = \alpha + \frac{f_A + f_B}{2} + X_{ij_i}(z_{j_i} - P_{j_i}). \tag{31}$$

The equilibrium factor prices P_A and P_B do not effect the equilibrium final wealth since investors do not trade away their initial factor endowments. The optimal choice for the speculative demand X_{ij_i} in (31) satisfies condition (24) for $j_i \in A$ and (30) for $j_i \in B$. The value of X_Z in (24) and (30) is given by the solution to the first order condition

$$\mathrm{E}_i\Big[U'\big(W_1\big)\big(z_j - P_j\big)\Big| z_{1j}, \hat{z}_{i2j}\Big] = 0 \tag{32}$$

where W_1 is given by (31). The equilibrium factor prices are given by the standard pricing equation

$$P_A = \mathrm{E}_i\left[\frac{U'(W_1)}{\mathrm{E}_i[U'(W_1)]} f_A\right] = \mathrm{E}_i\left[\frac{U'(W_1)}{\mathrm{E}_i[U'(W_1)]} f_B\right] = P_B \tag{33}$$

where W_1 is given by (31). Note that due to the symmetry of the idiosyncratic returns

$z_j - P_j$ in (17) and the demands X_{ij_i} in (24) and (30) the expectation in (33) is identical for all investors. Equation (33) is an explicit solution for the factor prices since the equilibrium wealth W_1 in this equation does not depend on P_A or P_B. Hence the existence of market clearing factor prices follows from the fact that the first order condition (32) has a unique solution.

Theorem 1 (Equilibrium)**.** *There exist factor prices $P_A = P_B$ such that the demands in Lemma 4 (b) are optimal. These factor prices are given by* (33)*.*

IV Short Sale Constraints

Assumption 1 (Short sale constraints)**.** *Assume a fraction λ of investors are not able to sell short so that $X_{ij} \geq 0$ for these investors.*

Similar to Section III assume that the equilibrium reveals all private signals and that prices are given by (14) with idiosyncratic components given by (15).

If prices P_j are given by (14) then the optimal wealth W_1 is given by (21) where $X_{ij_i} \geq 0$, $X_A \geq 0$ and $X_B \geq 0$ for the investors with short sale constraints. If an unconstrained investor i receives a signal $\hat{z}_{i2j}$ about firm $j = j_i$ then his demand for firm j_i according to (24) and (30) is given by

$$X_{j_i} = \begin{cases} \pm X_z^u & \text{if} \quad \hat{z}_{i2j} = \pm\sigma_z \quad \text{and} \quad \Big(j_i \in A \quad \text{or} \quad z_{1j}\hat{z}_{i2j} > 0\Big) \\ \pm(X_z^u - x^{u*}) & \text{if} \quad \hat{z}_{i2j} = \pm\sigma_z \quad \text{and} \quad \Big(j_i \in B \quad \text{and} \quad z_{1j}\hat{z}_{i2j} < 0\Big) \end{cases} \tag{34}$$

where $X_z^u > 0$ is the fraction of shares outstanding of firm j_i. If $P_A < 0$ and $P_B < 0$ then the firm-specific demand of a constrained investor is given by

$$X_{j_i} = \begin{cases} +X_{z_A}^c & \text{if} \quad \hat{z}_{i2j} = +\sigma_z \quad \text{and} \quad j_i \in A \\ +X_{z_B}^c & \text{if} \quad \hat{z}_{i2j} = +\sigma_z \quad \text{and} \quad j_i \in B \quad \text{and} \quad z_{1j} = +\sigma_z \\ +(X_{z_B}^c - x^{c*}) & \text{if} \quad \hat{z}_{i2j} = +\sigma_z \quad \text{and} \quad j_i \in B \quad \text{and} \quad z_{1j} = -\sigma_z \\ 0 & \text{if} \quad \hat{z}_{i2j} = -\sigma \end{cases} \tag{35}$$

where $X_z^c > 0$. The group of unconstrained investors can take the other side of the individual unconstrained trades in (34) without affecting their final wealth W_1 as described in (26). The group of constrained investors might not be able to take the other side of the individual trades in (35) due to their short sale constraints. However, we can assume without loss of generality that the unconstrained investors also take the other side of (35) since these investors are able to adjust their factor exposure separately.

Since the trades of the unconstrained investors in (34) reveal the same information as the demands of the constrained investors in (35) the idiosyncratic price components P_{z_j} are equivalent in both cases. This equivalence poses a minor problem since we might have $X_z^u \neq X_z^c$ and since the investors who take the other side of these trades can only condition their demands on the price. I will therefore assume that all investors observe before they start trading which firms are associated with constrained investors. This assumption is not strong since the equilibrium prices reveal this information. Alternatively it is possible to achieve an equivalent equilibrium by changing the way how the auctioneer chooses the equilibrium price so that he can also choose prices that do not fully clear the market.

Assume each unconstrained investor buys X_j shares of a particular firm j. Then the group of unconstrained investors buys a fraction of $(1-\lambda)X_j$ of the shares outstanding of firm j. Hence then unconstrained investors they can take the other side of the trades in (34) by submitting the demands

$$X_j(P_j) = \begin{cases} \pm\frac{X_z^u}{1-\lambda} & \text{if} \quad P_j = \alpha + P_A \mp (+\sigma_z + \frac{1}{2}q\sigma_z) \quad \text{and} \quad j \in \{A, B\} \\ \pm\frac{X_z^u}{1-\lambda} & \text{if} \quad P_j = \alpha + P_A \mp (-\sigma_z + \frac{1}{2}q\sigma_z) \quad \text{and} \quad j \in A \\ \pm\frac{X_z^u - x^{u*}}{1-\lambda} & \text{if} \quad P_j = \alpha + P_A \mp (-\sigma_z + \frac{1}{2}q\sigma_z) \quad \text{and} \quad j \in B \end{cases} \tag{36}$$

for firms with private signals received by unconstrained investors, where $X_j(P_j)$ is the number of shares that each unconstrained investor buys of firm j. Accordingly the unconstrained investors can take the other side of the trades in (35) by submitting the demands

$$X_j(P_j) = \begin{cases} -\frac{X_{z_A}^c}{1-\lambda} & \text{if} \quad P_j = \alpha + P_A + (\pm\sigma_z + \frac{1}{2}q\sigma_z) \quad \text{and} \quad j \in A \\ -\frac{X_{z_B}^c}{1-\lambda} & \text{if} \quad P_j = \alpha + P_A + (+\sigma_z + \frac{1}{2}q\sigma_z) \quad \text{and} \quad j \in B \\ -\frac{X_{z_B}^c - x^{c*}}{1-\lambda} & \text{if} \quad P_j = \alpha + P_A + (-\sigma_z + \frac{1}{2}q\sigma_z) \quad \text{and} \quad j \in B \end{cases} \tag{37}$$

for firms with private signals received by constrained investors. In addition to these trades investors might also want to adjust their factor exposures by trading equally weighted portfolios of firms in group A and B. Let $X_{A,\,X_{j_i}}^c$ be the number of shares per firm of an equally weighted portfolio of firms in group A that a constrained investor with the demand X_{j_i} given by (35) holds after trading. Define $X^c_{B,\, X_{j_i}}$, $X^u_{A,\, X_{j_i}}$ and $X^u_{B,\, X_{j_i}}$ accordingly. Then the average factor exposures for constrained investors that result from equally weighted portfolios are according to (19) and (20) given by

$$X^c_A = \frac{1}{2}\left(\frac{1}{2}X^c_{A,\,0} + \frac{1}{4}X^c_{A,\,X^c_{z_A}} + \frac{1}{8}X^c_{A,\,X^c_{z_B}} + \frac{1}{8}X^c_{A,\,X^c_{z_A}-x^{c*}}\right) \tag{38a}$$

$$X^c_B = \frac{1}{2}\left(\frac{1}{2}X^c_{B,\,0} + \frac{1}{4}X^c_{B,\,X^c_{z_A}} + \frac{1}{8}X^c_{B,\,X^c_{z_B}} + \frac{1}{8}X^c_{B,\,X^c_{z_A}-x^{c*}}\right) \tag{38b}$$

Accordingly the average factor exposures for unconstrained investors that result from equally weighted portfolios are given by

$$X^u_A = \frac{1}{2}\left(\frac{3}{4}X^u_{A,\,X^u_z} + \frac{1}{4}X^u_{A,\,X^u_z-x^{u*}}\right) \tag{39a}$$

$$X^u_B = \frac{1}{2}\left(\frac{3}{4}X^u_{B,\,X^u_z} + \frac{1}{4}X^u_{B,\,X^u_z-x^{u*}}\right) \tag{39b}$$

Since the trades in (34), (35), (36) and (37) add up to zero and since all investors are initially endowed with one share per firm market clearing requires

$$\lambda X^c_A + (1-\lambda)X^u_A = \frac{1}{2} \tag{40a}$$

$$\lambda X^c_B + (1-\lambda)X^u_B = \frac{1}{2} \tag{40b}$$

Adding the factor exposure that results from the information-based demands in (35) to the diversified portfolios in (38a) we obtain the average total factor exposures after trading for constrained investors:

$$\overline{X^c_A} = \frac{1}{2}\left(\frac{X^c_{A,\,0}}{2} + \frac{X^c_{A,\,X^c_{z_A}} + 2X^c_{z_A}}{4} + \frac{X^c_{A,\,X^c_{z_B}}}{8} + \frac{X^c_{A,\,X^c_{z_B}-x^{c*}}}{8}\right) \tag{41a}$$

$$\overline{X^c_B} = \frac{1}{2}\left(\frac{X^c_{B,\,0}}{2} + \frac{X^c_{B,\,X^c_{z_A}}}{4} + \frac{X^c_{B,\,X^c_{z_B}} + 2X^c_{z_B}}{8} + \frac{X^c_{B,\,X^c_{z_B}-x^{c*}} + 2(X^c_{z_B} - x^{c*})}{8}\right) \tag{41b}$$

The demands in (36) do not affect the final wealth of an unconstrained investor i. The information-based demands of the unconstrained investors in (34) cancel out within the group of unconstrained investors. Hence by adding the trades in (37) to the diversified portfolios in (39a) we obtain the average total factor exposures after trading for unconstrained investors:

$$\overline{X^u_A} = \frac{1}{2}\left[\frac{3}{4}\left(X^u_{A,\,X^u_z} - \frac{\lambda}{2}\frac{X^c_{z_A}}{1-\lambda}\right) + \frac{1}{4}\left(X^u_{A,\,X^u_z - x^{u*}} - \frac{\lambda}{2}\frac{X^c_{z_A}}{1-\lambda}\right)\right] \tag{42a}$$

$$\overline{X^u_B} = \frac{1}{2}\left[\frac{3}{4}\left(X^U_{B,\,X^u_z} - \frac{\lambda}{4}\frac{2X^c_{z_B} - x^{c*}}{1-\lambda}\right) + \frac{3}{4}\left(X^U_{B,\,X^u_z - x^{u*}} - \frac{\lambda}{4}\frac{2X^c_{z_B} - x^{c*}}{1-\lambda}\right)\right] \tag{42b}$$

Adding the total factor exposures after trading of the constrained investors in (41) to the total factor exposures after trading of the unconstrained investors in (42) we get:

$$\lambda\overline{X^c_A} + (1-\lambda)\overline{X^u_A} = \lambda X^c_A + (1-\lambda)X^u_A = \frac{1}{2} \tag{43a}$$

$$\lambda\overline{X^c_B} + (1-\lambda)\overline{X^u_B} = \lambda X^c_B + (1-\lambda)X^u_B = \frac{1}{2} \tag{43b}$$

Equations (43) shows that irrespective of how individual investors achieve their factor exposure, the average factor exposure always has to equal the average factor supply. Let

$\Delta(A,B)_{X_{i_j}}$ be the difference between the total exposure to factor f_A and f_B for an investor with the information-based demand X_{j_i}. For example

$$\Delta(A,B)^c_{X^c_{z_A}} = X^c_{A,\,X^c_{z_A}} + X^c_{z_A} - X^c_{B,\,X^c_{z_A}}.$$

Suppose

$$\Delta(A,B)^c_{X^c_{z_A}} - \frac{\Delta(A,B)^c_{X^c_{z_B}} + \Delta(A,B)^c_{X^c_{z_B}-x^{c*}}}{2} > 0 \tag{44}$$

so that the exposure of f_A over f_B for constrained investors with information-based demand $X^c_{z_A}$ dominates the average exposure of f_B over f_A for the remaining constrained investors with information-based demands. Subtracting (43b) from (43a) we get

$$\lambda\left(\frac{\Delta(A,B)^c_{X^c_{z_A}}}{4} + \frac{\Delta(A,B)^c_{X^c_{z_B}} + \Delta(A,B)^c_{X^c_{z_B}-x^{c*}}}{8}\right) \tag{45}$$

$$= \; -\left[\lambda\left(\frac{\Delta(A,B)^c_0}{2}\right) + (1-\lambda)\left(\frac{3\Delta(A,B)_{X^u_z}}{4} + \frac{\Delta(A,B)^u_{X^u_z - x^{u*}}}{4}\right)\right]$$

Applying (44) to (45) shows that if the group of constrained investors with information-based demands overweighs factor f_A over f_B then someone else in the economy has to underweigh factor f_A. But Lemma 2 shows that the remaining investors are only willing to underweigh factor f_A if

$$P_A > P_B.$$

Applying Lemma 2 again we see that all remaining investors want to overweigh f_B if

$P_A > P_B$. Hence we must have

$$\Delta(A, B) < 0$$

for all investors except for the constrained investors who buy $+X^c_{z_A}$ of firms $j \in A$. The constraints of these investors must be binding so that

$$X^c_{z_A} > \frac{1}{2} \quad \text{and} \quad X^c_{A,\, X^c_{z_A}} = 0.$$

Suppose an unconstrained investor i buys a fraction $X_{j_i} = X^u_z$ of the shares outstanding of firm $j = j_i \in A$ under the conditions described in (34). According to (42) the total exposure of this investor to factor f_A after trading is given by

$$X^u_{A,\, X^u_z} - \frac{\lambda}{2} \frac{X^c_{z_A}}{1 - \lambda}.$$

Since the investor is initially endowed with an exposure of 1/2 per factor and since he buys a fraction X^u_z of the shares outstanding of a firm in group A he can achieve his total exposure to f_A with the net trade

$$t^u_A = X^u_{A,\, X^u_z} - X^u_z - \frac{\lambda}{2} \frac{X^c_{z_A}}{1 - \lambda} - \frac{1}{2}.$$

Since the investor starts with an endowment of zero of firm j_i define the net trade for firm j_i as $t^u_z = X^u_z$. Table 1 shows these net trades for all investors and various values of the factor correlation ρ as defined in (5), the fraction of short-sell constrained investors λ and the probability of informative signals q. In these examples all investors have power utility

$U(x) = \frac{1}{1-\gamma}x^{1-\gamma}$ with $\gamma = 1.5$. The remaining parameter values are given by $\alpha = 100$, $\sigma_f = \sigma_z = 20$. The returns $R_A = \frac{\alpha}{\alpha+P_A} - 1$ and $R_B = \frac{\alpha}{\alpha+P_B} - 1$ are the expected returns of the mimicking portfolios for the factors f_A and f_B. As the table shows the expected returns of the illiquid firms in B exceed the returns of the liquid firms in A provided that $\rho < 1$, $\lambda > 0$ and that the probability of an informative signal q is sufficiently high. In these cases the constrained investors who observe private positive signals about liquid firms in A buy more than 50% of the shares outstanding of firms $j_i \in A$ ($t^c_{z_Z} > 0.5$) and sell their complete endowment of factor A ($t^c_A = -0.5$). Since the average investor who receives a positive signal about a firm in B trades less aggressively, the demand for the factor f_A that determines the dividends of the liquid firms exceeds the demand for the factor f_B that determines the dividends of the illiquid firms.

V Conclusion

In this paper I examine how the price impact of trading affects the cross-section of expected returns. I develop a model where a large number of investors trade a large number of assets. The dividends of these assets depend on systematic and on firm-specific components. Private information about the firm-specific components causes investors to impact prices when they trade. Investors impact the prices of liquid asset with higher information asymmetry more than prices of illiquid assets with lower information asymmetry. Investors can trade large diversified portfolios without price impact.

When investors speculate on their private information they trade liquid assets more aggressively then illiquid assets. Liquid assets exhibit therefore more trading volume then illiquid assets. An investor who overweighs a particular firm in his portfolio also automatically overweighs the systematic risk factors that determine the dividend of this firm. The investor can adjust his systematic risk exposure by short-selling a diversified portfolio that mimics these risk factors. All speculative trades in individual firms are therefore accompanied by opposite trades in diversified portfolios. As a result the speculative demand for idiosyncratic risk does not affect the demand for systematic risk. Since idiosyncratic risk is not priced and since investors can trade large portfolios without price impact, liquidity is irrelevant for expected returns in a frictionless market.

If some investors cannot short sell then these investors hold only long speculative positions of individual assets. Since the investors trade liquid assets more aggressively than illiquid assets they overweigh liquid assets in their portfolios. Since the investors cannot undue these trades by shorting factor mimicking portfolios they also overweigh systematic dividend risk factors of liquid assets. Hence other investors in the economy have to underweigh these risk factors. In the equilibrium the risk premia of factors that determine the dividends of liquid assets are therefore higher than the risk premia of comparable factors that determine the dividends of illiquid assets. Liquidity does not create additional systematic risk beyond the systematic dividend risk.

Appendix A: Proof of Lemma 2

Part one. Let $\tilde{X}_A = X_A + X_{i_j}$. The cost of a portfolio that consists of $\tilde{X}_A$ shares of each asset in group A and X_B shares of each asset in B is given by

$$C = \tilde{X}_A(\alpha + P_A) + X_B(\alpha + P_B)$$

Hence for a fixed total investment C the trade-off between shares in A and B is given by

$$X_B = \frac{C - \tilde{X}_A(\alpha + P_A)}{\alpha + P_B} \tag{46}$$

Plugging (46) in the the wealth equation (21) we get

$$\frac{\partial \mathrm{E}_i\left[U(W_1)\middle|\hat{z}_{ij2}\right]}{\partial \tilde{X}_A} = E_i\left[U'(W_1) \times \left(f_A - P_A - \frac{\alpha + P_A}{\alpha + P_B}(f_B - P_B)\right)\middle|\hat{z}_{ij2}\right] \tag{47}$$

Assume $P_A < P_B$. To prove the lemma we have to show that (49) is strictly positive at $\tilde{X}_A = X_B = X$ for any given values of X, X_0 and X_{j_i}. Recall that $f_A, f_B \in \{-\sigma, \sigma\}$. We have for $f_A = f_B = \pm\sigma$

$$\begin{aligned} f_A - P_A - \frac{\alpha + P_A}{\alpha + P_B}(f_B - P_B) &= \pm\sigma\left(1 - \frac{\alpha + P_A}{\alpha + P_B}\right) - P_A + P_B\frac{\alpha + P_A}{\alpha + P_B} \\ &= (\alpha \pm \sigma)\frac{P_B - P_A}{\alpha + P_B} \end{aligned}$$

and for $f_a = -f_b = \pm\sigma$

$$f_A - P_A - \frac{\alpha + P_A}{\alpha + P_B}(f_B - P_B) = \frac{\alpha(P_B - P_A) \pm \sigma(P_A + P_B + 2\alpha)}{\alpha + P_B}$$

Define

$$\bar{w} = X_0 - X(P_A + P_B) + X_{j_i}\big(z_{2j} - \mathrm{E}_{h\neq i}\big[z_{2j}\big|\hat{z}_{i2j}\big]\big)$$

Then we have

$$\begin{aligned}
\frac{\partial \mathrm{E}\big[U(W_1)\big|z_{2j}, \hat{z}_{i2j}\big]}{\partial \tilde{X}_A} \quad &= \quad U'(\bar{w} + 2HX\sigma) \times (\alpha + \sigma)\frac{P_B - P_A}{\alpha + P_B} \\
&+ \quad U'(\bar{w} - 2X\sigma) \times (\alpha - \sigma)\frac{P_B - P_A}{\alpha + P_B} \\
&+ \quad 2U'(\bar{w}) \times \alpha\frac{P_B - P_A}{\alpha + P_B} \qquad (48)
\end{aligned}$$

Since $U' > 0$, $\alpha + P_B > 0$, and $P_A < P_B$ the third term on the right hand sight of (48) is strictly positive. Since $\alpha > \sigma$ the sum of the first two terms is strictly positive.

Part two. Assume $P_A = P_B = P$. Then $E[U(W_1)]$ attains its maximum at some $\tilde{X}_A = X_B = X$ by part one of the lemma. Let

$$W_1 = X_0 + X(f_A + f_B - 2P) + X_{j_i}(z_{j_i} - P_{j_i})$$

Then we have

$$\frac{\partial \mathrm{E}_i\big[U(W_1)\big|\hat{z}_{ij2}\big]}{\partial X} = E_i\big[U'(W_1)\big] \times E\big[f_A + f_B - 2P\big] \qquad (49)$$

at $H = 0$. The result follows since $E[f_A] = E[f_B] = 0$.

Appendix B: Proof of Lemma 3

The idiosyncratic return $z_j - P_j$ for $j = j_{iB}$ is given by (17). Since $z_j - P_j$ is independent of f_A and f_B and since $E_i\left[z_j - P_{z_j}\middle|\hat{z}_{i2j}\right] > 0$ by (18a) we have

$$E_i\left[U\big(0\big)\middle|z_{1j} = \sigma_z, \hat{z}_{i2j} = \sigma_z, P_j = \alpha + P_A - \sigma_z + \frac{1}{2}q\sigma_z, \mathbf{P}\right]$$

$$< \quad E_i\left[U\big(X_Z\big)\middle|z_{1j} = \sigma_z, \hat{z}_{i2j} = \sigma_z, P_j = \alpha + P_A + \sigma_z + \frac{1}{2}q\sigma_z, \mathbf{P}\right]$$

and

$$\frac{\partial E_i\left[U\big(X_j\big)\middle|z_{1j} = \sigma_z, \hat{z}_{i2j} = \sigma_z, P_j = \alpha + P_A - \sigma_z + \frac{1}{2}q\sigma_z, \mathbf{P}\right]}{\partial X_j} > 0$$

for $X_j = 0$

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Table 1: **Equilibrium with Short Sale Constraints.** This table shows the equilibrium of Section IV for various values of the factor correlation ρ, the fraction of short sale constrained investors λ and the probability that a private signal is informative q. R_A and R_B are the expected returns of the factors f_A and f_B. The standard deviation of the factor mimicking portfolios is given by $\sigma_f = 20\%$. The net trade of factor f_A by unconstrained investors who buy X_{j_i} share outstanding of firm j_i is given by t^u_A. The net trade of firm j_i is given by $t^u_z = X_{j_i}$. The remaining trades are defined accordingly. Utility functions are given by $U(x) = -x^{-0.5}$.

	ρ	0	0	1	0	0	0
	λ	0	0.5	0.5	0.5	0.5	0.75
	q	0.2	0.2	0.75	0.5	0.75	0.75
	R_A (in %)	3.17	3.17	6.21	3.05	2.86	2.54
	R_B (in %)	3.17	3.17	6.21	3.13	3.10	2.92
$X_{j_i} = X^u_z$	t^u_z	0.32	0.32	1.45	0.88	1.53	1.54
	t^u_A	−0.32	−0.32	−1.45	−0.91	−1.63	−1.67
	t^u_B	0	0	0.10	−0.02	−0.06	−0.09
$X_{j_i} = X^u_z - x^{u*}$	t^u_z	0.01	0.01	0.14	0.05	0.14	0.15
	t^u_A	0	0	−0.12	−0.02	−0.04	−0.09
	t^u_B	−0.01	−0.01	−0.02	−0.05	−0.15	−0.18
$X_{j_i} = X^c_{z_A}$	$t^c_{z_A}$		0.32	1.22	0.68	1.00	0.97
	t^c_A		−0.32	−0.50	−0.50	−0.50	−0.50
	t^c_B		0	−0.50	−0.02	−0.04	−0.07
$X_{j_i} = X^c_{z_B} - x^{c*}$	$t^c_{z_B}$		0.01	0.13	0.05	0.11	0.10
	t^c_A		0	−0.10	−0.02	−0.04	−0.09
	t^c_B		−0.01	−0.02	−0.05	−0.11	−0.14
$X_{j_i} = X^c_{z_B}$	$t^c_{z_B}$		0.32	1.22	0.69	1.20	1.01
	t^c_A		0	−0.50	−0.03	−0.08	−0.13
	t^c_B		−0.32	−0.50	−0.50	−0.50	−0.50
$X_{j_i} = 0$	t^c_A		0	−0.14	−0.02	−0.05	−0.10
	t^c_B		0	0.13	0	−0.01	−0.04