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Are stocks desirable in tax-deferred accounts? ☆

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Abstract

Existing literature suggests that, in order to maximize the tax benefit of retirement accounts, investors should follow a "pecking order" location rule of placing highly taxed assets (e.g., bonds) in a tax-deferred account and lightly taxed assets (e.g., stocks) in a taxable account. Empirical evidence, however, documents that a large number of investors violate this rule. In this paper, we show that such violations can be optimal for risk-averse investors who face portfolio constraints. In particular, while the strategy of placing bonds in the tax-deferred account maximizes the expected *level* of tax benefit, it may lead to volatile benefits under different realizations of stock returns. By holding a similar portfolio in both accounts, investors can achieve a more balanced growth in the two accounts, minimize the likelihood of violating the constraints in the future and hence "smooth" the *volatility* of the tax benefit. For some risk-averse investors, this smoothing motive can lead to the observed violation of the pecking order location rule. Our model predicts that such violations are more likely when future tax benefits are more volatile, which can occur, for example, when: (i) the tax rate differential across assets increases over time due either to tax law changes or to tax bracket changes for investors; (ii) asset returns are more volatile; and (iii) investors anticipate large future liquidity needs.

JEL classification: G11
Keywords: Portfolio selection; Tax-deferred accounts; Borrowing and short-selling constraints

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1. Introduction

In recent years, tax-deferred retirement accounts (IRA, Roth-IRA, 401(k) and 403(b)) have enjoyed a dramatic increase in popularity.[1] According to academic literature[2] and popular advice,[3] investors who have access to both a taxable and a tax-deferred account should follow a "pecking order" location rule of placing highly taxed assets (e.g., bonds) in a tax-deferred account and lightly taxed assets (e.g., stocks) in a taxable account. Stocks should be held in a tax-deferred account only if the entire taxable account is already fully invested in stocks. However, this pecking order location rule seems to be violated in practice, since many investors place stocks in the tax-deferred account while holding bonds in the taxable account.[4]

In this paper we investigate how the limit on borrowing or short-selling assets in both accounts can induce a preference towards holding stocks in the tax-deferred account, hence violating the pecking order location rule. In order to isolate the effect of portfolio constraints, we consider the simplest possible model of asset allocation with two accounts and two assets (one risky stock and one riskless bond). We are able to obtain an analytic solution for the optimal dynamic portfolio of an investor with logarithmic preferences. Despite its streamlined form, this model has surprisingly rich implications for the impact of portfolio constraints on asset location decisions.

The main outcome of the model is the determination that portfolio constraints can lead to the violation of the pecking order location rule. Intuitively, the opportunity to invest in a tax-deferred account generates an implicit *tax subsidy*, whose magnitude is controlled by the portfolio the investor chooses. In an *unconstrained* problem, Huang (2005) shows that the optimal tax-deferred strategy is to maximize the tax subsidy by placing only the assets with the highest tax rate in the tax-deferred account. In this case, both the tax-deferred strategy and the resulting tax subsidy are state-independent. In a *constrained* problem, however, the possibility of incurring future binding constraints makes the tax subsidy state-dependent. As a consequence, when choosing an optimal location strategy, in addition to the motive of maximizing the *expected* tax subsidy, risk-averse investors will also choose a strategy that minimizes the subsidy's *variability*. This "smoothing" of the tax subsidy across possible states of nature is the major driving force behind the preference for holding an interior mix of stocks and bonds in both the taxable and tax-deferred accounts.

More specifically, our analysis shows that violations of the pecking order can be obtained *only* in a dynamic setting, implying that a single-period logic can be flawed when used to describe and understand empirical evidence from household portfolio holdings. In a dynamic setting, the possibility of future binding portfolio constraints makes the future tax subsidy volatile. The desire to smooth the *future* tax subsidy competes with the *current* preference towards holding the highest-taxed asset in the tax-deferred account. Violations of the pecking

[1] Using data from the Survey of Consumer Finances, Bergstresser and Poterba (2004) report that 49.1% of households held assets in tax-deferred retirement accounts in 2001, up from 30.7% in 1989. Moreover, in 2001, the median household with both tax-deferred and taxable financial assets had 60% of its financial assets in a tax-deferred account.

[2] See, for example, Ameriks and Zeldes (2001), Amromin (2003), Auerbach and King (1983), Black (1980), Bodie and Crane (1997), Dammon et al. (2004), Huang (2005), Poterba et al. (2004), Shoven and Sialm (2003), and Tepper (1981).

[3] See, for example, the *Vanguard* website "How to be a tax-savvy investor?" as an example of popular advice.

[4] Poterba and Samwick (2003) document that 48% of households who own bonds in their taxable accounts also hold stocks in their tax-deferred accounts and 41.6% of households who own stocks in their tax-deferred accounts also own bonds in their taxable accounts. Both of these findings are in violation of the pecking order location rule. Barber and Odean (2003), using brokerage account data, document that investors seem to be understanding the benefits of locating the highest-taxed asset in the tax-deferred account but, at the same time, find that the same investors do not fully take advantage of such tax-avoidance strategies.

order location rule occur when the former effect dominates the latter. Empirically, our model predicts that such violations are more likely, for example, when (i) the tax rate differential across assets increases over time due either to tax law changes or to tax bracket changes for investors; (ii) asset returns are more volatile; and (iii) investors anticipate large future liquidity needs.

The problem of allocating wealth between accounts with differential tax treatments has been covered extensively in the finance literature. In the context of optimal asset composition within corporate pension accounts, Tepper and Affleck (1974), Black (1980), and Tepper (1981) use a static tax-arbitrage argument to show that, if a corporation can issue debt and hold stocks in the corporate account, it is always optimal to fully fund the pension account and to hold only bonds in it. Auerbach and King (1983) point out that this arbitrage extends to individuals making choices between taxable and tax-deferred retirement accounts. Huang (2005) generalizes the result to a dynamic setting under the assumption that investors are allowed to borrow and short-sell in their taxable accounts, and confirms the location result of preferring bonds in the tax-deferred account. Dammon et al. (2004) incorporate borrowing and short-selling constraints–together with other realistic features–in a lifetime portfolio decision problem with both accounts and focus on liquidity needs as the driving force behind possible violations. Using numerical simulations they conclude that, while theoretically possible, under reasonable parameter values liquidity concerns are not sufficient to induce a preference for stock in the tax-deferred account.

Our main contribution to this literature is to identify the role of the tax subsidy, implied by the existence of a tax-deferred account, as a crucial component for the determination of the optimal portfolio with both taxable and tax-deferred accounts. Current studies have focused on the *level* of the tax subsidy, which is maximized through the above-mentioned tax-arbitrage strategies. We argue that, under constraints, the *volatility* of the tax subsidy is crucial in understanding how constraints affect optimal portfolio choices. In a dynamic setting we show that, due to the desire of risk-averse investors to smooth risk across states, violations of the pecking order can be obtained when the desire to smooth the *future* tax subsidy dominates the *current* tax preference for a particular asset. The existing literature focused on the level of tax subsidy and relied on the existence of substantial liquidity needs in order to generate a simultaneous holding of stocks and bonds in both accounts. Our results are derived within a parsimonious model of portfolio choice with constraints and are obtained without invoking investors' liquidity needs or penalties for withdrawing from the tax-deferred account. To the best of our knowledge, we are the first to isolate and explore the role of volatility of the tax subsidy in explaining violations of the pecking order location rule.

The rest of the paper proceeds as follows. Section 2 presents the model used to study the constrained portfolio problem with two accounts. Section 3 presents the solution in the last period, and Section 4 contains the optimal portfolio in the first period and develops empirical predictions emerging from our model. Section 5 concludes. Appendix A contains the proofs for all propositions.

2. The model

We consider a two-period portfolio problem in which an investor faces borrowing and short-selling constraints in both taxable and tax-deferred accounts. The investment horizon begins at time $t=0$. At each time $t=0$, 1, there are two financial assets available for investment: a risky stock and a riskless bond. The investor can decide how much of each asset to hold in each account. At time $t=2$ the investor liquidates his portfolio and consumes the realized wealth.

Without loss of generality, we will treat the tax-deferred account as a Roth-type tax-exempt account.[5]

2.1. Asset returns

The only source of uncertainty is represented by the stochastic evolution of the return on the risky asset. Let $\tilde{r}_t$ be the random, before-tax gross return on the stock between time t and time $t+1$. At each time $t=0$, 1, we assume that such a random variable can take only values r_{Ht} and r_{Lt} with equal probability. The return distributions of $\tilde{r}_0$ and $\tilde{r}_1$ are independent over time. These will be the returns investors would realize by holding a stock in a tax-deferred account. For tractability purposes, we ignore optimal capital gains realizations and assume that all capital gains and losses are realized immediately and taxed at a rate $\tau_{\mathrm{S}t}$ which reflects the un-modeled optimal tax realization strategy in the risky asset.[6] When the stock is held in the taxable account, its after-tax random return $\tilde{R}_t$ has the following realizations:

$$R_{it} = r_{it} - \tau_{\mathrm{S}t}(r_{it} - 1), \qquad i = H, L, \quad t = 0, 1. \tag{1}$$

Similarly, we denote by r_{ft} the (non-stochastic) before-tax return on the risk-free bond between period t and period $t+1$. This is the return on the bond when held in a tax-deferred account. If the bond is held in a taxable account, its after-tax return will be

$$R_{ft} = r_{ft} - \tau_{\mathrm{B}t}(r_{ft} - 1), \qquad t = 0, 1, \tag{2}$$

where $\tau_{\mathrm{B}t}$ is the tax rate on bond income at time t. For future reference, it is useful to introduce a hypothetical return $R_{f_s t}$, defined as the net return on the bond if its income were to be taxed at the tax rate of the stock, i.e.,

$$R_{f_s t} = r_{ft} - \tau_{\mathrm{S}t}(r_{ft} - 1), \qquad t = 0, 1. \tag{3}$$

We capture, in a reduced form, the impact of optimal capital gains realizations on the stock by assuming that $\tau_{\mathrm{S}t} \le \tau_{\mathrm{B}t}$ for all t, and we rule out obvious arbitrage possibilities by imposing the requirement that $r_{Lt} < r_{ft} < r_{Ht}$, $t=0$, 1. Finally, to insure positive demand for the risky asset by risk-averse investors, we require that its expected return exceeds the risk-free rate, i.e., $1/2(r_{Ht} + r_{Lt}) > r_{f,t}$, $t=0$, 1. Since $\tau_{\mathrm{S}t} \le \tau_{\mathrm{B}t}$, the same condition also holds for the after-tax returns, i.e., $1/2(R_{Ht} + R_{Lt}) > R_{f,t}$, $t=0$, 1.

2.2. The portfolio problem

Since the return distribution is changing over time, to fully separate the effects of this change in the investment opportunity set from the impact of the co-existence of taxable and tax-deferred accounts, we assume that the investor has logarithmic utility. It is well known (see Merton, 1971) that for such preferences, the unconstrained dynamic optimal portfolio of an

[5] In practice, investors can decide whether to contribute pre-tax money and be taxed at withdrawal (as in Traditional IRA and 401(k) plans) or to contribute after-tax money and withdraw tax-free (e.g. Roth–IRA). As long as the marginal income tax rate of the investor changes deterministically over time, the economic implications of the two forms of contribution are identical.

[6] See for example, the "effective tax rate" in Constantinides (1983) and Huang (2005).

investor who has access to only one account is myopic. Hence, any deviation from the myopic solution can be attributed to the presence of multiple accounts and the borrowing and short-selling constraints.

For simplicity, we assume the investor derives utility only from terminal wealth at time $t=2$, and ignore the decision to contribute wealth to the tax-deferred account.[7] We denote by W_t^{T} and W_t^{D} the wealth available at time t in, respectively, the taxable and tax-deferred accounts. For convenience, we also define the quantities W_t and ϕ_t as, respectively, the total nominal wealth, and the fraction of total nominal wealth held in the tax-deferred account:

$$W_t = W_t^{\mathrm{T}} + W_t^{\mathrm{D}}, \quad \phi_t = \frac{W_t^{\mathrm{D}}}{W_t^{\mathrm{T}} + W_t^{\mathrm{D}}}. \tag{4}$$

A generic portfolio is represented by the pair $(\alpha_t^{\mathrm{T}}, \alpha_t^{\mathrm{D}})$, where α_t^{T} and α_t^{D} are, respectively, the fraction of taxable wealth W_t^{T} and tax-deferred wealth W_t^{D} invested in the stock at time t. The presence of borrowing and short-selling constraints necessitates that $\alpha_t^{\mathrm{T}} \in [0, 1]$ and $\alpha_t^{\mathrm{D}} \in [0, 1]$, at $t=0, 1$.

Given initial endowments of wealth W_0^{T} and W_0^{D} in the two accounts, the investor's problem is to choose the portfolio strategy over time that maximizes the expected utility of his terminal wealth. We assume that, at time $t=2$, investors liquidate the entire tax-deferred account at no penalty for terminal consumption. Hence, the objective function for time $t=0$, and 1 can be written as:

$$J_t(W_t^{\mathrm{T}}, W_t^{\mathrm{D}}) \equiv \max_{\{\alpha_t^{\mathrm{T}}, \alpha_t^{\mathrm{D}}\}} E_t[\log(\widetilde{W}_2^T + \widetilde{W}_2^D)], \qquad t = 0, 1, \tag{5}$$

where $E_t[\cdot]$ denotes expectation at time t and

$$\widetilde{W}_{t+1}^T = W_t^{\mathrm{T}}(\alpha_t^{\mathrm{T}}(\widetilde{R}_t - R_{f,t}) + R_{f,t}), \tag{6}$$

$$\widetilde{W}_{t+1}^D = W_t^{\mathrm{D}}(\alpha_t^{\mathrm{D}}(\widetilde{r}_t - r_{f,t}) + r_{f,t}), \tag{7}$$

$$\alpha_t^{\mathrm{T}}, \alpha_t^{\mathrm{D}} \in [0, 1], \tag{8}$$

where Eqs. (6) and (7) describe the evolution of taxable and tax-deferred wealth, and Eq. (8) represents the borrowing and short-selling constraints in both accounts.

2.3. Single-account portfolios

The solution for the single-account version of the above problem without borrowing and short-selling constraints is well known (see, for example, Ingersoll, 1987). Specifically, when investors

[7] Since we do not consider intermediate consumption, there is no need to explicitly model withdrawals from the tax-deferred account. In Section 4.2 we consider a simple extension of the model that allows for such withdrawals.

Table 1
Model quantities

Symbol	Definition	Reference
r_{it}	Before-tax return on stock at time t and state $i=H, L$	
r_{ft}	Before-tax return on bond at time t	
τ_{St}	Tax rate on stock at time t	
τ_{Bt}	Tax rate on bond at time t	
R_{it}	After-tax return on stock at time t and state $i=H, L$	Eq. (1)
R_{ft}	After-tax return on bond at time t	Eq. (2)
R_{f_St}	Hypothetical return on bond if taxed at τ_{St}	Eq. (3)
W_t^{T}	Wealth in the taxable account at time t	
W_t^{D}	Wealth in the tax-deferred account at time t	
W_t	Nominal wealth at time t	Eq. (4)
ϕ_t	Relative size of the tax-deferred account at time t	Eq. (4)
α_t^{T}	Fraction of taxable wealth invested in stock at time t	
α_t^{D}	Fraction of tax-deferred wealth invested in stock at time t	
α_{tM}^{T}	Optimal taxable–account-only portfolio at time t	Eq. (9)
α_{tM}^{D}	Optimal tax-deferred–account-only portfolio at time t	Eq. (9)
z_t	Tax subsidy at time t	Definition 1

have access to only a taxable (tax-deferred) account, the optimal portfolio is myopic and given by $\alpha_{tM}^{\mathrm{T}}(\alpha_{tM}^{\mathrm{D}})$, where

$$\alpha_{tM}^{\mathrm{T}}=\frac{R_{ft}}{2}\left[\frac{1}{R_{ft}-R_{Lt}}-\frac{1}{R_{Ht}-R_{ft}}\right],\quad \alpha_{tM}^{\mathrm{D}}=\frac{r_{ft}}{2}\left[\frac{1}{r_{ft}-r_{Lt}}-\frac{1}{r_{Ht}-r_{ft}}\right]. \tag{9}$$

For simplicity, we assume that the parameters are such that the above portfolios strictly satisfy the constraints in Eq. (8), i.e., $\alpha_{tM}^{\mathrm{T}}, \alpha_{tM}^{\mathrm{D}}\in(0, 1)$. These single-account optimal portfolios are the “growth optimal portfolios” in Merton (1971). It is possible to show that $\alpha_{tM}^{\mathrm{T}}>\alpha_{tM}^{\mathrm{D}}$, i.e., investors hold more stocks if they are investing in a taxable account. Considering the stock excess return over the bond in the two accounts, the stock has a better risk-return profile on an after-tax basis than on a pre-tax basis, since taxation dampens the volatility of the stock while reducing the expected return of both assets.

2.4. The effective tax subsidy

By allowing an investor to shield assets’ income from taxation in tax-deferred accounts, the government is effectively providing him with a *tax subsidy*. Intuitively, such a subsidy is the compensation that an investor would require to renounce his right to invest in a tax-deferred account. We formalize the concept of tax subsidy in the following definition.[8]

Definition 1. Let $J_t(W_t^{\mathrm{T}}, W_t^{\mathrm{D}})$ be the indirect utility function, defined in Eq. (5), of an investor with both taxable and tax-deferred accounts. The tax subsidy per dollar held in the tax-deferred account is the value $z_t(W_t^{\mathrm{T}}, W_t^{\mathrm{D}})$, such that

$$J_t(W_t^{\mathrm{T}},W_t^{\mathrm{D}})=J_t(W_t^{\mathrm{T}}+(1+z_t)W_t^{\mathrm{D}},0). \tag{10}$$

[8] Poterba (2004) uses the term “equivalent taxable wealth” to identify a similar quantity in a non-stochastic setting.

The righthand side, $J_t(W_t^{\mathrm{T}}+(1+z_t)W_t^{\mathrm{D}}, 0)$, is the value function for an investor with access to *only* a taxable account. Therefore, an investor is indifferent about deferring taxes on assets held in his retirement account and receiving instead a lump-sum tax subsidy of z_t dollars for each dollar in his retirement account.

For ease of reference, we report in Table 1 a list of the model's relevant quantities.

3. Optimal portfolios at time $t=1$

We solve the problem backwards using dynamic programming. At time 1, we can explicitly derive the optimal portfolio. The following proposition summarizes the solution, with details of the proof provided in Appendix A.

Proposition 1. *The optimal portfolio (α_1^{T*}, α_1^{D*}) at time $t=1$ is given by*

$$\alpha_1^{\mathrm{T}*} = \begin{cases} \alpha_{1M}^{\mathrm{T}}\left(1+\dfrac{\phi_1}{1-\phi_1}\dfrac{r_{f1}}{R_{f1}}\right) & \text{if } 0 \le \phi_1 \le \phi_1^{\mathrm{A}} \\ 1 & \text{if } \phi_1^{\mathrm{A}} < \phi_1 \le \phi_1^{\mathrm{B}} \\ 1 & \text{if } \phi_1^{\mathrm{B}} < \phi_1 < 1 \end{cases}, \tag{11}$$

$$\alpha_1^{\mathrm{D}*} = \begin{cases} 0 & \text{if } 0 < \phi_1 \le \phi_1^{\mathrm{A}} \\ 0 & \text{if } \phi_1^{\mathrm{A}} < \phi_1 \le \phi_1^{\mathrm{B}} \\ \alpha_{1M}^{\mathrm{D}}\left(1+\dfrac{1-\phi_1}{\phi_1}\dfrac{R_{f_s1}}{r_{f1}}\right)-\dfrac{1-\phi_1}{\phi_1}(1-\tau_{S1}) & \text{if } \phi_1^{\mathrm{B}} < \phi_1 \le 1 \end{cases}, \tag{12}$$

where ϕ_1 is defined in Eq. (4)*, α_{1M}^T and α_{1M}^D are defined in Eq.* (9)*, and $\phi_1^A \le \phi_1^B$ are constants given in Eqs.* (A14) and (A18).

The above proposition shows that investors appear to follow a "pecking order" location rule: higher-taxed assets (e.g., bonds) are placed in the tax-deferred account first, followed by lower-taxed assets (e.g. stocks). The solution consists of three branches defined over three different regions of the relative size of the tax-deferred account ϕ_1. We will refer to the case $0 \le \phi_1 \le \phi_1^{\mathrm{A}}$ as "Region A"; to the case $\phi_1^{\mathrm{A}} < \phi_1 \le \phi_1^{\mathrm{B}}$ as "Region B"; and to the case $\phi_1^{\mathrm{B}} < \phi_1 \le 1$ as "Region C." If the tax-deferred account is small (Region A), investors place only bonds in the tax-deferred account, and the remaining desired holdings of bonds and all stocks are placed in the taxable account. For a medium-sized tax-deferred account (Region B), investors hold only bonds in the tax-deferred account and only stocks in the taxable account. When the size of the tax-deferred account is sufficiently large (Region C), the desired amount of bond holding is not sufficient to fill the tax-deferred account, and investors place some stocks in the tax-deferred account while they hold only stocks in the taxable account. *In a single-period problem, a simultaneous holding of stocks and bonds in both accounts is never possible.*

This pecking order location result is consistent with the findings in Huang (2005) in an unconstrained setting. Using a tax-arbitrage argument, she shows that, in the absence of portfolio constraints, placing bonds in the tax-deferred account always yields a higher tax subsidy and hence investors should hold *only* bonds in such account. With constraints, however, investors will stop adding to their bond positions in the tax-deferred account when the only way to offset an undesirable exposure to tax-deferred bonds is to *sell short* bonds in the taxable account. As a result, they will start holding stocks in the tax-deferred account whenever their taxable account is invested entirely in stocks, confirming the pecking order location rule.

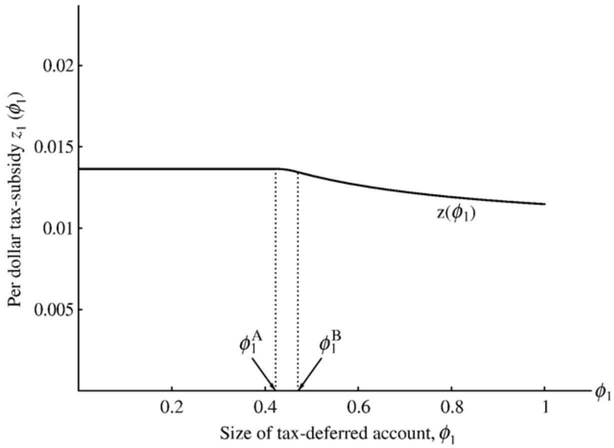

Fig. 1. Tax subsidy. The figure reports the per dollar tax subsidy $z_1(\phi_1)$, (Eq. (14)) as a function of the relative size ϕ_1 of the tax-deferred account. Parameters used: $\tau_{S1}=25\%$, $\tau_{B1}=35\%$, $r_{f1}=1.04$, $r_{H1}=1.5$, $r_{L1}=0.7$. These parameters imply a pre-tax mean excess return on the stock of 6% and a volatility of 40%, yielding a Sharpe ratio of 15%.

The optimal portfolio *allocation* in Region A can be understood using a replication argument. Since a tax-deferred dollar invested in bonds yields r_{f1}, while a dollar of taxable bonds yields R_{f1} in the same period, each dollar of tax-deferred bonds has the same payoff as r_{f1}/R_{f1} dollars of taxable bonds. In region A, only bonds are placed in the tax-deferred account. The two-account problem can hence be thought of as equivalent to a single-account problem in which investors have access to only a taxable account with total wealth $\widehat{W}_1^{\mathrm{T}}=W_1^{\mathrm{T}}+W_1^{\mathrm{D}}(r_{f1}/R_{f1})$, where the second term can be thought of as a "non-tradable" portion of the investor's total wealth invested in the taxable portfolio that *replicates* his tax-deferred holdings (in this case, bonds). The investor can use the total wealth $\widehat{W}_1^{\mathrm{T}}$ to determine the optimal level of stock holding when constraints do not bind. Specifically, since an investor with access to only a taxable account optimally invests α_{1M}^{T} of his wealth in stocks, the optimal *dollar* holding when he also has access to a tax-deferred account is $\alpha_{1M}^{\mathrm{T}}\widehat{W}_1^{\mathrm{T}}$, provided this quantity satisfies the constraint $\alpha_{1M}^{\mathrm{T}}\widehat{W}_1^{\mathrm{T}}<W_1^{\mathrm{T}}$. This constraint holds for smaller tax-deferred accounts, and sets the boundary for Region A. The percentage holding of $\alpha_1^{\mathrm{T}*}$ in Eq. (11) can be derived by simplifying the expression $\alpha_{1M}^{\mathrm{T}}\widehat{W}_1^{\mathrm{T}}/W_1^{\mathrm{T}}$.

If the desired stock holding $\alpha_{1M}^{\mathrm{T}}\widehat{W}_1^{\mathrm{T}}$ is larger than the total taxable wealth W_1^{T}, the need to hold stocks in the tax-deferred account arises. However, from the solutions (11)–(12), we note that investors do not immediately hold stocks in the tax-deferred account once the capacity of the taxable account is reached. In Region B, investors hold only bonds in the tax-deferred account and only stocks in the taxable account. The reason behind this portfolio choice is that stocks are less attractive in the tax-deferred account, as previously discussed.

In Region C, when the desired stock holding $\alpha_{1M}^{\mathrm{T}}\widehat{W}_1^{\mathrm{T}}$ is considerably larger than the taxable wealth W_1^{T}, investors start holding stocks in the tax-deferred account. The allocation problem becomes considerably more complicated. Without knowledge of the exact mix between tax-deferred stocks and bonds, the argument used in Region A does not apply directly since one cannot calculate the equivalent taxable wealth. However, since we showed that the optimal location strategy in this case is to hold only stocks in the taxable account, we can derive the optimal allocation by *changing the viewpoint* of the investor. Taking the tax-deferred account as the "reference" account, we can represent taxable stock and bond returns through a combination of tax-deferred stocks and bonds. The detailed derivation is provided in Appendix A, where we show that, for Region C, we can transform the two-account problem into a "tax-deferred account-only" problem with an equivalent tax-deferred wealth of $\widehat{W}_1^{\mathrm{D}}=W_1^{\mathrm{T}}(R_{f_s1}/r_{f1})+W_1^{\mathrm{D}}$. As before, the quantity $W_1^{\mathrm{T}}(R_{f_s1}/r_{f1})$ can be viewed as a "non-tradable" portfolio of tax-deferred stocks and bonds, corresponding to the effective payoff of all the taxable stocks. Since α_{1M}^{D} is the optimal portfolio for a "tax-deferred account-only" problem, the optimal dollar holding of stocks is $\alpha_{1M}^{\mathrm{D}}\widehat{W}_1^{\mathrm{D}}$. To determine the optimal stock holding in the tax-deferred account, we need to subtract the stock exposure inherited from the holding in the taxable account. It can be shown that the effective tax-deferred exposure for each dollar invested in taxable stock is $(1-\tau_{\mathrm{S}1})$ dollars. Therefore, investors optimally hold $\alpha_{1M}^{\mathrm{D}}\widehat{W}_1^{\mathrm{D}}-W_1^{\mathrm{T}}(1-\tau_{\mathrm{S}1})$ dollars of tax-deferred stocks. Dividing this quantity by the wealth W_1^{D} in the tax-deferred account yields the expression for the percentage holding of $\alpha_1^{\mathrm{D}*}$ in Eq. (12).

Given the solution in Proposition 1, the following Corollary explicitly derives an expression for the indirect utility function and for the effective tax subsidy at time 1.

Corollary 1. *The indirect utility function $J_1(W_1^T, W_1^D)$ for the constrained problem*(5) *is*

$$J_1(W_1^{\mathrm{T}}, W_1^{\mathrm{D}}) = \log(A_1) + \log(W_1^{\mathrm{T}} + (1+z_1(\phi_1))W_1^{\mathrm{D}}), \tag{13}$$

where

$$z_1(\phi_1) = \begin{cases} \dfrac{r_{f1}}{R_{f1}}-1 & \text{if } 0 \le \phi_1 \le \phi_1^{\mathrm{A}} \\ \dfrac{1}{\phi_1}\left(\dfrac{B_1}{A_1}\sqrt{\left(1+\phi_1\left(\dfrac{r_{f1}}{R_{H1}}-1\right)\right)\left(1+\phi_1\left(\dfrac{r_{f1}}{R_{L1}}-1\right)\right)}-1\right) & \text{if } \phi_1^{\mathrm{A}} < \phi_1 \le \phi_1^{\mathrm{B}} \\ \dfrac{1}{\phi_1}\left(\dfrac{C_1}{A_1}\left(1+\phi_1\left(1-\dfrac{R_{f_s1}}{r_{f1}}\right)\right)-1\right) & \text{if } \phi_1^{\mathrm{B}} < \phi_1 \le 1 \end{cases} \tag{14}$$

is the tax subsidy defined in Eq. (10)*, and A_1, B_1, C_1 are constants defined in Eqs.* (A20)–(A22).

Fig. 1 illustrates how the tax subsidy $z_1(\phi_1)$ changes with the relative size ϕ_1 of the tax-deferred account. The parameters are chosen to reflect realistic values. Return numbers are similar to historical averages of asset returns at an annual frequency. Specifically, we set the risk free rate to 4%, the individual asset volatility to 40%, the Sharpe ratio of assets to 15%, which implies a 10% expected return (or a 6% risk premium) for the stock. The tax rates for stocks and bonds are set at 25% and 35%, respectively. The bond tax rate is chosen to be the same as the rate for the highest tax bracket in 2005. Sialm (2006) finds considerable variation in the effective tax rate on equity, ranging from approximately 40% in the early 1950s to about 5% in 2004. Moreover, even in the same year, the effective tax rate can vary significantly across assets depending on the composition of dividends and capital gains components.[9]

In Region A, the tax subsidy is constant and equal to $(r_{f1}/R_{f1})-1$. From our previous discussion, we know that one dollar invested in bonds in the tax-deferred account has the same return as r_{f1}/R_{f1} dollars of taxable bonds. Hence, the privilege of being able to invest in the tax-deferred account is equivalent to receiving from the government z_1 dollars of tax subsidy for each

[9] Although the choice of 25% tax rate for stocks is somewhat arbitrary, in unreported result, we have experimented with tax rates on stocks ranging from 10% to 30%, and obtained qualitatively similar results.

dollar in the tax-deferred account. Clearly, this statement is valid only if the investor holds all bonds in the tax-deferred account and is unconstrained, which is true for Region A.

In other regions, however, investors are constrained and may hold stocks in the tax-deferred account. Since stocks are taxed less heavily than bonds, placing them in the tax-deferred account yields lower benefit. Since the quantity ϕ_1 is a function of the portfolio chosen at time 0, the tax subsidy itself will depend endogenously on the current period portfolio choices. The endogeneity of the tax subsidy $z_1(\phi_1)$ significantly complicates the portfolio choice problem at time zero, to which we now turn.

4. Optimal portfolios at time $t=0$

In the previous section we have shown that, at time 1, investors follow a pecking order location rule which places highly taxed assets in the tax-deferred account in order to maximize the effective tax subsidy received. At time 0, there is an additional concern: since their current period portfolio choices also influence the dynamics of the future tax subsidy, risk-averse investors may also desire to minimize the variability of the future tax subsidy. In this section we show that this smoothing motive can induce violations of the pecking order location rule.

We start by rewriting the portfolio problem (5) at time 0 using the value function in Eq. (13):

$$J_0(W_0^{\mathrm{T}}, W_0^{\mathrm{D}}) \equiv \max_{\{\alpha_0^{\mathrm{T}},\alpha_0^{\mathrm{D}}\}} E_0[\log(A_1) + \log(\widetilde{W}_1^T + (1 + z_1(\phi_1))\widetilde{W}_1^D)]. \tag{15}$$

The above problem is complicated by the fact that $z_1(\phi_1)$ is a function of ϕ_1 which, in turn, depends on the portfolio choice (α_0^{T}, α_0^{D}) at time $t=0$. To understand the main intuition behind location decisions, let us consider first the reduced-form problem

$$\widehat{J}_0(W_0^{\mathrm{T}}, W_0^{\mathrm{D}}) = \max_{\{\alpha_0^{\mathrm{T}},\alpha_0^{\mathrm{D}}\}} E_0[\log(A_1) + \log(\widetilde{W}_1^T + (1 + z)\widetilde{W}_1^D)], \tag{16}$$

where the endogenous $z_1(\phi)$ is replaced by an exogenous variable $z>0$. The following lemma summarizes the solution for this reduced-form problem.

Lemma 1. *The optimal solution (α_0^{T*}, α_0^{D*}) to the reduced-form problem in Eq.* (16) *under constraints* (6)–(8) *always satisfies the pecking order location rule, that is, either (i) $\alpha_0^{T*} \in [0, 1]$ and $\alpha_0^{D*}=0$, or (ii) $\alpha_0^{T*}=1$ and $\alpha_0^{D*} \in [0, 1]$.*

The proof of this lemma relies on a replication argument similar to the one used in Proposition 1 and is given in Appendix A. Lemma 1 confirms that investors should follow pecking order location rules when the future tax subsidy is constant. From Eq. (14), however, the subsidy is constant only in Region A, and a decreasing function of ϕ_1 otherwise. Moreover, the tax subsidy $z_1(\phi_1)$ depends on current period portfolio choices only through the realizations of ϕ_1. Using the definition of ϕ_1 in Eq. (4), we can rewrite the value function $J_0(W_0^{\mathrm{T}}, W_0^{\mathrm{D}})$ in Eq. (15) as

$$J_0(W_0^{\mathrm{T}}, W_0^{\mathrm{D}}) \equiv \max_{\{\alpha_0^{\mathrm{T}},\alpha_0^{\mathrm{D}}\}} E_0[\log(A_1) + \log(\widetilde{W}_1) + \log(1 + \phi_1 z_1(\phi_1))]. \tag{17}$$

The function $J_0(\cdot)$ is concave in ϕ_1. Hence, everything else being equal, a portfolio that smoothes the relative size of future tax-deferred account ϕ_1 across states improves expected utility. Clearly, any portfolio that smoothes ϕ_1 across different states also smoothes the tax subsidy $z_1(\phi_1)$. The following definition formalizes the concept of “smoothing portfolio”.

Definition 2. A smoothing portfolio at time t is a portfolio (α_t^{T}, α_t^{D}) that equalizes the relative size ϕ_{t+1} of the tax-deferred account across states, i.e., $\phi_{H,t+1}=\phi_{L,t+1}$.

The next lemma characterizes the structure of the smoothing portfolio at time 0.

Lemma 2. *Any portfolio (α_0^T, α_0^D) that satisfies*

$$\alpha_0^{\mathrm{D}} = \frac{r_{f0}(1-\tau_{\mathrm{S0}})\alpha_0^{\mathrm{T}}}{R_{f0} + (r_{f0}-1)(\tau_{\mathrm{B0}}-\tau_{\mathrm{S0}})\alpha_0^{\mathrm{T}}} \tag{18}$$

is a smoothing portfolio.

Notice that, if $\alpha_0^{\mathrm{T}}\in(0, 1)$, then Lemma 2 implies that $\alpha_0^{\mathrm{D}}\in(0, 1)$. Hence, the smoothing portfolio in Lemma 2 implies interior holdings of stocks and bonds in both accounts and clearly violates the pecking order location rule. In general, when making their optimal asset location decisions, investors trade off the pecking order preference, which maximizes the expected level of tax subsidy, for the smoothing preference, which reduces the volatility of the tax subsidy. To fully understand the interaction between these two forces, we consider, in the following section, the special case in which there is no tax preference between stocks and bonds at time zero. This case allows us to isolate and study the smoothing effect on portfolio choices.

4.1. Optimal portfolios without tax preferences in the current period

Consider the case in which stocks and bonds are taxed at the same rates at time 0, $\tau_{\mathrm{S0}}=\tau_{\mathrm{B0}}$, but have different tax rates at time 1, $\tau_{\mathrm{S1}}<\tau_{\mathrm{B1}}$. This case is interesting since, without tax preferences in the current period, the location effect on the optimal portfolio will only be due to differential taxation of the two assets at time 1. This allows us to isolate the impact of the future tax subsidy on current location preferences. In the following proposition we derive the optimal portfolio in this case.

Proposition 2. *Let $\tau_{S0}=\tau_{B0}$. The optimal portfolio (α_0^{T*}, α_0^{D*}) at time $t=0$ is given by*

$$\alpha_0^{\mathrm{T}*} = \begin{cases} \alpha_{0M}^{\mathrm{T}} - \dfrac{\phi_0 r_{f1}}{(1-\phi_0)(1-\tau_{\mathrm{S0}})R_{f1}}\delta_{\mathrm{A}} & \text{if } 0 \le \phi_0 \le \phi_0^{\mathrm{A}} \\ \alpha_{0M}^{\mathrm{T}} & \text{if } \phi_0^{\mathrm{A}} < \phi_0 \le \phi_0^{\mathrm{B}}, \\ \alpha_{0M}^{\mathrm{T}} - \dfrac{\phi_0 r_{f1}}{(1-\phi_0)(1-\tau_{\mathrm{S0}})R_{f_s1}}\delta_{\mathrm{C}} & \text{if } \phi_0^{\mathrm{B}} < \phi_0 < 1 \end{cases} \tag{19}$$

$$\alpha_0^{\mathrm{D}*} = \begin{cases} \alpha_{0M}^{\mathrm{D}} + \delta_{\mathrm{A}}, & \text{if } 0 < \phi_0 \le \phi_0^{\mathrm{A}} \\ \alpha_{0M}^{\mathrm{D}} & \text{if } \phi_0^{\mathrm{A}} < \phi_0 \le \phi_0^{\mathrm{B}}, \\ \alpha_{0M}^{\mathrm{D}} + \delta_{\mathrm{C}} & \text{if } \phi_0^{\mathrm{B}} < \phi_0 \le 1 \end{cases} \tag{20}$$

where α_{0M}^T and α_{0M}^D are defined in Eq. (9),

$$\phi_0^{\mathrm{A}} = \frac{\phi_1^{\mathrm{A}}}{(1-\phi_1^{\mathrm{A}})\frac{r_{f0}}{R_{f0}} + \phi_1^{\mathrm{A}}}, \quad \phi_0^{\mathrm{B}} = \frac{\phi_1^{\mathrm{B}}}{(1-\phi_1^{\mathrm{B}})\frac{r_{f0}}{R_{f0}} + \phi_1^{\mathrm{B}}}, \tag{21}$$

and where, for any given ϕ_0, the quantities δ_A and δ_C belong to the closed and convex sets

$$\delta_{\mathrm{A}}\in[\delta_{\mathrm{A}}^{-}(\phi_0), \delta_{\mathrm{A}}^{+}(\phi_0)], \quad \delta_{\mathrm{C}}\in[\delta_{\mathrm{C}}^{-}(\phi_0), \delta_{\mathrm{C}}^{+}(\phi_0)], \tag{22}$$

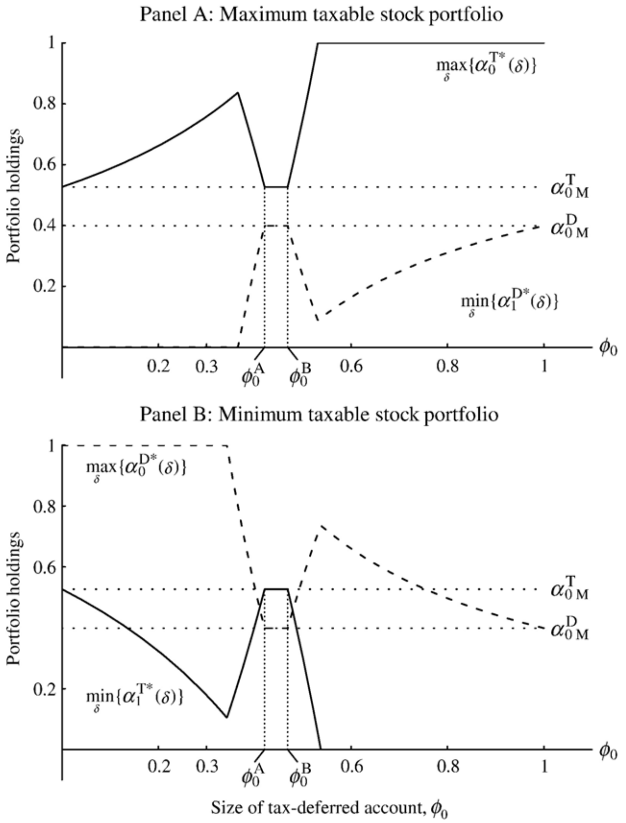

Fig. 2. Optimal portfolio holding without current tax preferences. The solid line in each figure reports, as a function of ϕ_0, the highest (Panel A) and lowest (Panel B) possible optimal stock in the taxable account (i.e., α_0^{T*} from Proposition 2). The dashed line reports the corresponding lowest (Panel A) and highest (Panel B) possible optimal stock in the tax-deferred account (i.e., α_0^{D*} from Proposition 2). The dotted horizontal lines represent the time 0 single-account portfolios α_{0M}^T and α_{0M}^D defined in Eq. (9). Parameters used: $\tau_{S1}=25\%$, $\tau_{B1}=35\%$, $r_{f1}=1.04$, $r_{H1}=1.5$, $r_{L1}=0.7$. These parameters imply a pre-tax mean excess return on the stock of 6% and a volatility of 40%, yielding a Sharpe ratio of 15%.

whose boundaries $\delta_A^-(\phi_0)<0<\delta_A^+(\phi_0)$, $\delta_C^-(\phi_0)<0<\delta_C^+(\phi_0)$ *are given by Eqs.* (A36), (A37), (A59), and (A60)*, respectively.*

An illustration of the solution is given in Fig. 2. The solid lines represent the upper (panel A) and lower (panel B) bounds of the optimal portfolio in taxable stocks, α_0^{T*}, as a function of ϕ_0. The upper (lower) bound is obtained by using Eq. (19) with $\delta_A=\delta_A^+(\phi_0)$ and $\delta_C=\delta_C^+(\phi_0)$ ($\delta_A=\delta_A^-(\phi_0)$ and $\delta_C=\delta_C^-(\phi_0)$). The dashed lines represent the corresponding lower (panel A) and upper (panel B) bounds of the optimal portfolio in tax-deferred stocks, obtained by using Eq. (20). The dotted horizontal lines represent the optimal portfolio α_{0M}^T (or α_{0M}^D) when investors have access to only a taxable (or tax-deferred) account. Since $\delta_A=\delta_C=0$ is always feasible, $(\alpha_{0M}^T, \alpha_{0M}^D)$ is always one of the optimal portfolios. Clearly, this portfolio violates the pecking order rule as $\alpha_{0M}^T, \alpha_{0M}^D \in (0, 1)$.

Interestingly, the portfolio $(\alpha_{0M}^T, \alpha_{0M}^D)$ implies $\phi_{H1}=\phi_{L1}$, i.e., it is one of the "smoothing" portfolios that we identified in Lemma 2. As we have discussed earlier, everything else being equal, the concavity of the value function implies that smoothing the relative size of the future tax-deferred account ϕ_1 improves utility. Although in general it is hard to keep everything else equal as we change portfolios in both accounts, we can show that, in our current specific setting, the separability between $\widetilde{W}_1$ and ϕ_1 in the value function (17) and the lack of current period tax preferences make it possible to indeed "keep everything else equal". As a result, the optimal (α_{0M}^{T}, α_{0M}^{D}) yields "complete smoothing", i.e., $\phi_{H1}=\phi_{L1}$. While the optimality of this complete smoothing is clearly specific to our setting, the intuition that a reduction in the volatility of ϕ_1 improves utility is robust.

Another interesting feature of the solution is that it is general not unique. We can in fact construct additional optimum by "perturbing" the smoothing portfolio (α_{0M}^{T}, α_{0M}^{D}). For example, when $\phi_0<\phi_0^{\mathrm{A}}$, we can increase tax-deferred stock holding by δ_{A} while at the same time reducing taxable stock holding by the amount indicated in Eq. (19) in order to maintain the desired risk exposure. As we have discussed above, in the current setting, we are able to smooth ϕ_1 while keeping everything else equal. However, smoothing is only relevant if the resulting tax subsidy z_1 (ϕ_1) can change with ϕ_1, depending on the realization of stock returns. In Appendix A, we show that when $\phi_0<\phi_0^{\mathrm{A}}$, under the optimal portfolio, the future relative sizes of the tax-deferred account, ϕ_{H1} and ϕ_{L1} are always in Region A. Eq. (14) indicates that the future tax subsidy $z_1(\phi_1)$ is constant in this case. Hence, for $\phi_0<\phi_0^{\mathrm{A}}$, smoothing is not necessary, and deviations from the smoothing portfolio yield the same utility as long as we balance the change in taxable portfolio with the appropriate offsetting change in the tax-deferred account.

The main finding in this subsection is that, when there are no preferences towards holding bonds in the tax-deferred account in the current period, the desire to smooth the future tax subsidy can generate violations of the pecking order location preference. Given the surprising nature of this result, in the next subsection we provide a simple numerical example to illustrate the main intuition.

4.1.1. An illustrative example

Let us consider an investor who faces a two-period investment problem as the one described in Section 2. For illustrative purposes we can think of each of the two periods as a year. The parameters of the problem are the same as those used in Fig. 2.

Assume the investor has a nominal wealth of 100 dollars, of which 40 dollars are invested in the tax-deferred account and the remaining in the taxable account. The relative size of the tax-deferred account is hence $\phi_0=40/100=0.4$. The optimal smoothing portfolio (α_{0M}^{T}, α_{0M}^{D}) in Proposition 2 is equal to (52.69%, 39.90%), which corresponds to \$31.61 and \$15.96 of stocks in the taxable and the tax-deferred account respectively. Clearly, this portfolio requires an interior mix of stocks and bonds in both accounts and thus violates the pecking order location rule.

For comparison purpose, we construct the best portfolio that satisfies the pecking order rule. That is, we solve the optimization problem in Eq. (15) under the constraint that either $\alpha_0^{\mathrm{T}}=1$ or $\alpha_0^{\mathrm{D}}=0$. The "optimal pecking order portfolio" thus constructed is $\alpha_0^{\mathrm{T}'}=88.19\%$ and $\alpha_0^{\mathrm{D}'}=0\%$, which corresponds to \$52.91 of stocks in the taxable account and only bonds in the tax-deferred account.

To understand the optimality of the smoothing portfolio, we focus on the value function in Eq. (17), and calculate the evolution of wealth $\widetilde{W}_1$ and of the relative size of the tax-deferred account ϕ_1 for both the optimal and the pecking order portfolios at time 1. First, the future wealth $\widetilde{W}_1$ in high and low states are \$121.647 and \$89.913 for the optimal portfolio, and \$121.655 and \$89.907 for the pecking order portfolio. We can verify that, under both portfolios, the value of the term $E_0[\log(\widetilde{W}_1)]$ in Eq. (17) is identical (and equal to $1/2(\log\widetilde{W}_{H1}+\log\widetilde{W}_{L1})=4.64998$). Hence, in this specific example, as we change from the pecking order portfolio to the smoothing portfolio, we are able to keep everything else equal *except* the relative size of the tax-deferred account ϕ_1 and the corresponding tax subsidy $z_1(\phi_1)$. In particular, the optimal portfolio yields

$\phi_{H1}=\phi_{L1}=0.4023$, and $z_1(\phi_{H1})=z_1(\phi_{L1})=0.01365$, and hence is a smoothing portfolio according to Definition 2. On the other hand, the pecking order portfolio yields $\phi'_{H1}=0.3420$, $\phi'_{L1}=0.4627$, with corresponding tax subsidy $z_1(\phi'_{H1})=0.01365=z_1(\phi_{H1})$, and $z_1(\phi'_{L1})=0.01350<z_1(\phi_{L1})$. Note that the subsidy received under the smoothing portfolio stochastically dominates the subsidy received under the pecking order portfolio. Since the value function (17) is concave in ϕ_1, and $E_0[\log(\widetilde{W}_1)]$ is identical under both portfolio strategies, the smoothing portfolio maximizes the investor's expected utility. This example confirms that the desire to smooth the future tax subsidy can generate violations of the pecking order location preference in the current period.

4.2. Optimal portfolios with tax preferences in the current period

In general, the tax rates on stocks and bonds differ at time 0. When bonds are taxed more heavily than stocks in the current period, a tax-arbitrage argument suggests that, if investors ignore the future dynamics of the tax subsidy, they should follow the pecking order location rule in order to maximize the expected tax subsidy. If, instead, investors are concerned about the volatility of the future tax subsidy, the results in Section 4.1 indicate that they should hold the "smoothing" portfolio that equalizes the subsidy across states, hence violating the pecking order. The optimal solution would then involve a trade off between the current-period pecking order motive and the future-period smoothing motive. The violation of the pecking order is more likely if the smoothing motive dominates, which can occur either if the pecking order motive is weak or if the smoothing motive is strong.

Despite the streamlined nature of our model, obtaining the general solution in analytical form is a surprisingly challenging task. Since our main goal is to understand the driving forces behind the desirability of holding stocks in the tax-deferred account, we forgo the search for a closed-form solution. Instead, based on the results obtained so far, we derive conjectures on assets' and investors' characteristics that may lead to violations of the pecking order location rule, and verify these conjectures by numerically solving the general model. The structure of the problem allows us to form three conjectures for the determinants of the pecking order violations.

Our first conjecture is that violations are more likely if the tax rate differential across stocks and bonds increases over time. While a smaller difference in current-period tax rates decreases the pecking order motive, a larger future tax rate differential increases the volatility of the future tax subsidy and hence increases the smoothing motive.

Our second conjecture is that violations of the pecking order are more likely when asset returns are more volatile. The rationale behind this conjecture is that the pecking order rule generally leads to the emergence of substantially different portfolios in the two accounts. As a consequence, when returns are more volatile, the growth in the two accounts is more unbalanced. The relative size of the tax-deferred account varies considerably across different states. Given the relationship between the tax subsidy and the size of the tax-deferred account in Fig. 1, we expect future tax subsidies to be more volatile for riskier asset returns. Therefore, the smoothing motive is stronger and violations are more likely.

Our third conjecture is that the anticipation of future liquidity shocks can lead to the violation of the pecking order location rule. The need to withdraw from the tax-deferred account can be thought of as a more stringent constraint on portfolio strategies. Since constraints make the tax subsidy more volatile, liquidity needs will strengthen the smoothing motive.

To understand the impact of liquidity shocks on portfolio choices, we modify our original model to introduce the possibility of a liquidity shock at time 1 in the simplest possible way. Specifically, we assume that, with probability λ, the investor loses a fixed portion of his original wealth, qW_0. For simplicity, we treat this wealth loss as an exogenous event. If withdrawing from the tax-deferred account is necessary to meet the liquidity needs, investors incur a penalty equal to a fraction η of the amount withdrawn. The value function in Eq. (15) can be modified as follows:

$$J_0(W_0^{\mathrm{T}}, W_0^{\mathrm{D}}) \equiv \max_{\{\alpha_0^{\mathrm{T}}, \alpha_0^{\mathrm{D}}\}} E_0[(1-\lambda)J_1(W_1^{\mathrm{T}}, W_1^{\mathrm{D}}) + \lambda J_1(W_{q1}^{\mathrm{T}}, W_{q1}^{\mathrm{D}})], \tag{23}$$

$$W_{q1}^{\mathrm{T}} = \begin{cases} W_1^{\mathrm{T}} - qW_0 \\ 0 \end{cases}, \quad W_{q1}^{\mathrm{D}} = \begin{cases} W_1^{\mathrm{D}}, & \text{if } W_1^{\mathrm{T}} \geq qW_0 \\ W_1^{\mathrm{D}} - \dfrac{1}{1-\eta}(qW_0 - W_1^{\mathrm{T}}), & \text{if } W_1^{\mathrm{T}} < qW_0 \end{cases}. \tag{24}$$

We also require that $q < 1 - \eta$, to insure that the investor always has sufficient wealth to meet the liquidity needs at time 1. The assumption that the liquidity shock is a percentage of the original wealth W_0 instead of time 1 wealth W_1 rules out the dependence of the liquidity shock on stock market returns. An alternative definition of a liquidity shock that depends on W_1 yields similar results.

With the three previously stated conjectures in mind, we numerically solve the general model and, in Fig. 3, we plot the optimal portfolio at time 0 as a function of the relative size ϕ_0 of the tax-deferred account. The solid (dashed) lines correspond to the optimal portfolio in the taxable (tax-deferred) account. First, to confirm the impact of tax rate differential on location decisions, we fix the tax rates for future periods at $\tau_{\mathrm{S1}} = 25\%$ and $\tau_{\mathrm{B1}} = 35\%$, set the current period tax rate for stocks to equal the future rate, $\tau_{\mathrm{S0}} = 25\%$, and vary the current period bond tax rate from $\tau_{\mathrm{B0}} = 25.1\%$ to 35% from panel A to C. Introducing a difference between tax rates on assets at time 0 removes the multiplicity of solutions illustrated in Fig. 2, since the current period pecking order preference for stocks in the taxable account breaks the tie in future smoothing benefits. Panel A of Fig. 3 resembles panel A of Fig. 2, which corresponds to the portfolio that holds the maximum amount of stocks in the taxable account among all optimal portfolios. As we can see from the figure, there is a considerable region of relative sizes of the tax-deferred account ϕ_0 in which the investor simultaneously holds stocks and bonds in both accounts. As we gradually increase τ_{B0}, the shape of the optimal portfolio is preserved, with significant violations for medium-sized tax-deferred accounts in panel B. But if τ_{B0} is large enough, the current period pecking order motive dominates. In particular, this effect is manifest in panel C where no violation occurs.

Comparing panels B and D of Fig. 3 confirms our second conjecture that a violation of pecking order is more likely when asset volatility is higher. In panel B, the stock has a volatility of 40%, while in panel D, the volatility is 30%. To have a meaningful comparison between these two scenarios, we require that the Sharpe ratio of the stock remains constant at a level of 15% per period. All tax rates are the same across the two panels. In panel B (high volatility) we observe substantial violations of the pecking order rule, while there are no violations in a low-volatility environment (panel D).

Panel E of Fig. 3 confirms our conjecture that anticipation of liquidity shocks can induce violations of the pecking order. This panel considers the same case analyzed in panel B with the addition of a negative liquidity shock of size $q = 20\%$, occurring with a probability $\eta = 30\%$ and withdrawal penalty $\eta = 0.1$. When the wealth in the taxable account is sufficient to cover the liquidity shock (i.e. when ϕ_0 is less than about 0.8) the optimal portfolio is similar to the one in panel B, except for a slight reduction in the amount of stocks held in both accounts.[10] For smaller

[10] The reduction in overall stock holding is driven by the assumption that the liquidity shock is independent of stock returns. Hence, the investor increases his bond holding to prepare for the potential liquidity shock.

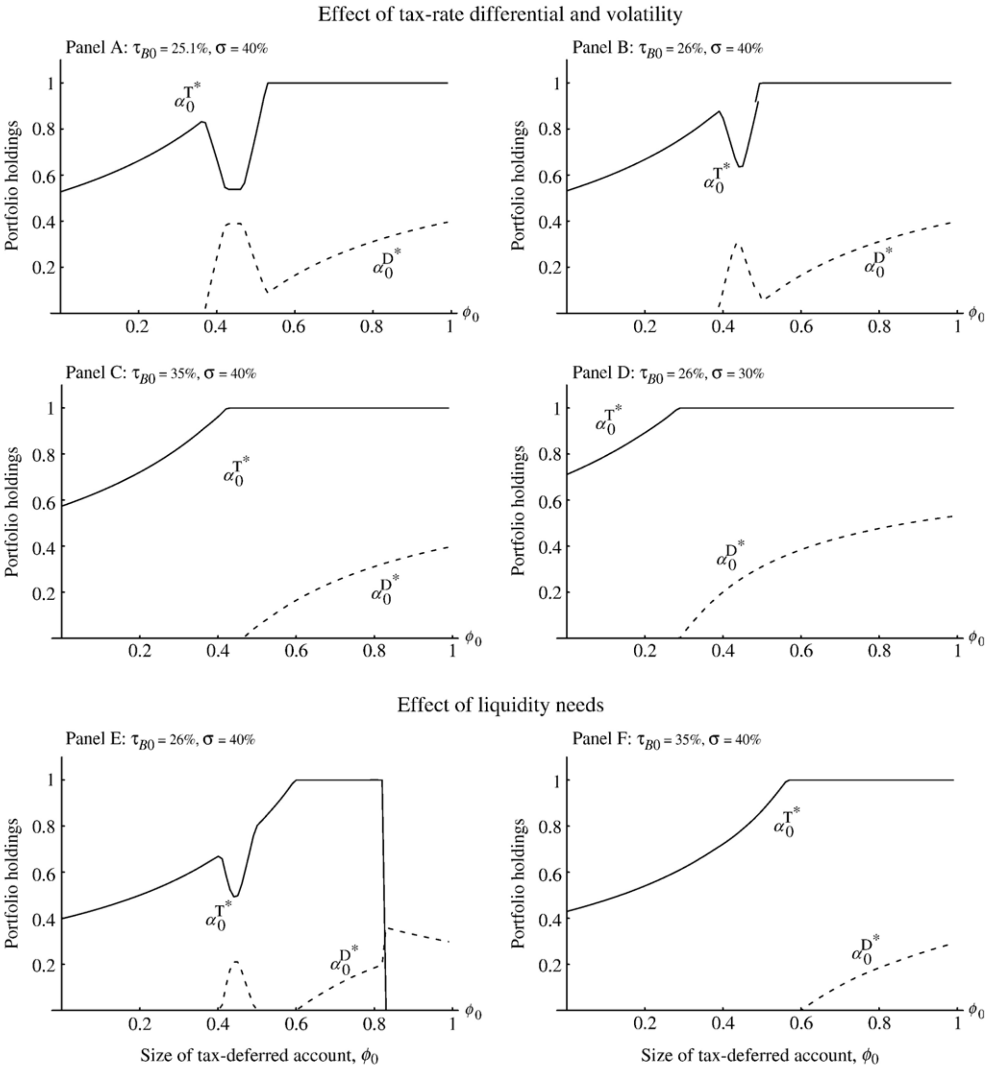

Fig. 3. Optimal portfolio holding with current tax preferences. The solid lines report, as a function of ϕ_0, the optimal stock holdings in the taxable account (α_0^{T*}). The dashed lines report the optimal stock holdings in the tax-deferred account (α_0^{D*}). Parameters used: $\tau_{S0}=25\%$, $\tau_{S1}=25\%$, $\tau_{B1}=35\%$, $r_{f0}=r_{f1}=1.04$. In all the panels other than D, $r_{H0}=r_{H1}=1.5$ and $r_{L0}=r_{L1}=0.7$, which imply a mean return of 10% and a volatility of $\sigma=40\%$. In panel D, $r_{H0}=r_{H1}=1.385$ and $r_{L0}=r_{L1}=0.785$, which imply a mean of 8.5% and a volatility of $\sigma=30\%$. In panels E and F the liquidity shocks have size $q=0.2$, occur with probability $\lambda=30\%$ and entail a withdrawal penalty $\eta=0.1$. The Sharpe ratio for the stock is constant at 15% in all the panels.

taxable wealth ($\phi_0>0.8$), however, taxable stock holdings drop to zero while tax-deferred stock holdings are positive, indicating a violation of the pecking order rule. Since the liquidity shock is independent of stock returns, placing bonds in the taxable account in those cases minimizes the expected withdrawal from the tax-deferred account in the event of a liquidity shock.

To assess the relative importance of the tax rate spread between stocks and bonds and of liquidity needs in generating violations of the pecking order, in panel F we increase the current period tax rate on the bond to 35%. This panel is equivalent to panel C, with the addition of liquidity needs. Given the stronger pecking order motive in the current period, the panel shows that, even substantial liquidity shocks are not sufficient to overturn the preference for bonds in the tax-deferred account. This finding is consistent with the results in Dammon et al. (2004) who find that, in the absence of tax-rate variation over time, liquidity motives are not sufficient to induce sizeable violations.

4.3. Empirical implications

Our model generates empirical conjectures that can potentially be tested in the data. In this section, we briefly discuss how the desirability of holding stocks in the tax-deferred account is affected by (i) the evolution of tax rates on assets, (ii) the volatility of assets, and (iii) the presence of future liquidity needs.

4.3.1. Evolution of tax rates

As observed in Section 4.2, the spread between tax rates on bonds and stocks over time determines the likelihood of observing violations of the pecking order location rule. The smaller the tax spread in the current period relative to the next period, the more likely it is to observe a preference for placing stocks in the tax-deferred account.

The literature has documented drastic tax rate changes across households' income groups, across different classes of income (dividend vs. capital gains), and over time. For example, Sialm (2006) shows that, over the period 1927–2004, the statutory marginal tax rate on dividends has ranged from 15% to 94% for households in the highest tax bracket and from 0.5% to 39% for households with income below 100,000 dollars (measured in 2004 consumer prices). On the other hand, the statutory marginal tax rate on long term capital gains has ranged from 12.5% to 35% for households in the highest tax bracket and from 0.5% to 28% for households with income below 100,000 dollars. Since the tax rate on bonds is similar to that on dividends and the tax rate on stocks is a mixture of those on dividends and capital gains, the above statistics suggest that the spread between the tax rate on bonds and stocks varies substantially over time and that such spread increases with households' income level.

Given this evidence, it is reasonable for investors to anticipate changes in tax rate spreads over time. In particular, if investors expect an increase in the future spread, they are more likely to prefer stocks in the tax-deferred account. On an individual level, since the tax rate spread between stocks and bonds generally increases with income level, we expect to observe more violations for younger or lower-income households who anticipate migration into higher-income tax brackets.

4.3.2. Volatility of returns

In Section 4.2, we also show that it is more likely that the pecking order location rules will be violated when the stock return is more volatile. That is, the more volatile the stock return is, the more likely it is for investors to prefer stocks in the tax-deferred account.

This result may appear counterintuitive. It is well known from the seminal work of Constantinides (1983) that more volatile stocks have lower effective tax rates, since the tax-timing option of postponing capital gains and realizing capital losses is more valuable for such assets. Therefore, higher volatility, by reducing the effective tax rate on stocks, should increase the chance that the stock is located in the taxable account.

To reconcile the two predictions, note that, in our model, we take the effective tax rate as fixed when we consider the impact of asset volatility. Hence, our result is complementary to the existing result by pointing out an alternative channel through which the volatility of an asset can affect its preferred location in an investor's portfolio.

In summary, taking as given the effective tax rate of an asset, investors are more likely to place higher volatility stocks in the tax-deferred account. Detecting this effect empirically, however, can be tricky due to the offsetting effect that volatility has on lowering the effective tax rate of stocks. Carefully disentangling these two effects will be important for understanding the net effect of asset volatility.

4.3.3. Liquidity needs

Liquidity needs and penalties for withdrawing from the tax-deferred account are usually invoked as necessary conditions for the observed violation of the pecking order rule (see, for example, Dammon et al., 2004). Our analysis shows that liquidity needs exacerbate future borrowing and short-selling constraints, leading to a more volatile future tax subsidy. The desire to smooth the future tax subsidy is the primitive force behind the possible preference for placing stocks in the tax-deferred account.

We predict that investors anticipating large future liquidity needs are more likely to display a preference for holding stocks in the tax-deferred account. Consistent with our prediction, both Barber and Odean (2003), using portfolio data on individual brokerage accounts, and Amromin (2003), using data from the Survey of Consumer Finances, find that liquidity needs and labor income shocks help explain some of the violation of the pecking order rule.

5. Conclusion

The opportunity to invest in tax-sheltered accounts, as well as the existence of constraints on borrowing and short-selling, increases considerably the complexity of the already-challenging task of investing for retirement.

Building on the concept of effective tax subsidy generated by the possibility of investing in a tax-deferred account, this paper analyzes the effect of borrowing and short-selling constraints on the optimal portfolio strategies. After solving explicitly for the optimal dynamic portfolio of an investor with logarithmic preferences, we identify two forces that emerge as determinants of the optimal asset location decision. The first force is the well-understood desire to hold the highest-taxed asset in the tax-deferred account (the pecking order location rule), in order to maximize the level of the tax subsidy. The second offsetting force is the desire to hold similar portfolios in the two accounts in order to smooth the volatility of the tax subsidy, which is caused by the possibility of binding future-period constraints. The analytical solution of the model identifies the smoothing motive as the driving force behind the desirability of placing stocks in the tax-deferred account.

A comparative static analysis of our model suggests that the violation of the preference for holding bonds in the tax-deferred account is more likely when: (i) the tax rate differential across assets increases over time due either to tax law changes or to tax bracket changes for investors; (ii) asset returns are more volatile; and (iii) investors expect large liquidity events. We leave the task of formally implementing a test of these predictions to future research.

Appendix A. Proofs

Proof of Lemma 1. The proof is by contradiction. Let $(\alpha_0^T, \alpha_0^D) \in (0, 1) \times (0, 1)$ be a generic portfolio with simultaneous holdings in stocks and bonds in both accounts and consider a perturbation $(\alpha_0^T+\delta^T, \alpha_0^D-\delta^D)$ of it with $\delta^T, \delta^D>0$. Starting with an initial wealth $W_0=W_0^T+W_0^D$ and given an exogenous z, the expected utility under the perturbed portfolio is

$$E_0[u(\widetilde{W}_1^T+z\widetilde{W}_1^D+W_0^T(\tilde{R}_1-R_{f1})\delta^T-W_0^Dz(\tilde{r}_1-r_{f1})\delta^D)]. \tag{A1}$$

Let (x_{S1}, y_{S1}) be a portfolio of stocks and bonds in the TA account that replicates the transformed return $z\tilde{r}_1$ in a stock in the TD account. Such a portfolio solves the following system of equations

$$zr_H=x_{S1}R_H+y_{S1}R_{f0} \tag{A2}$$

$$zr_L=x_{S1}R_L+y_{S1}R_{f0}, \tag{A3}$$

i.e.,

$$x_{S1}=\frac{z}{1-\tau_{S1}}, \quad y_{S1}=z\left(\frac{r_{f1}}{R_{f1}}-\frac{R_{f_s1}}{R_{f1}}\frac{1}{1-\tau_{S1}}\right). \tag{A4}$$

The cost z_{S1} of replicating $z\tilde{r}_1$ is, hence,

$$z_{S1}\equiv x_{S1}+y_{S1}=z\left(\frac{r_{f1}}{R_{f1}}-\left(\frac{R_{f_s1}}{R_{f1}}-1\right)\frac{1}{1-\tau_{S1}}\right). \tag{A5}$$

Similarly, let (x_{B1}, y_{B1}) be a portfolio of stocks and bonds in the TA account that replicates the transformed return zr_f in a bond in the TD account,

$$zr_{f1}=x_{B1}R_H+y_{B1}R_{f0} \tag{A6}$$

$$zr_{f1}=x_{B1}R_L+y_{B1}R_{f0}. \tag{A7}$$

The solution to the above system is $x_{B1}=0$ and $y_{B1}=z(r_{f1}/R_{f1})$, with a corresponding replication cost

$$z_{B1}\equiv x_{B1}+y_{B1}=z\frac{r_{f1}}{R_{f1}}. \tag{A8}$$

Substitute Eqs. (A2)–(A3) and (A6)–(A7) in Eq. (A1) and choose

$$\delta^T=\frac{\phi_0}{1-\phi_0}x_{S1}\delta^D>0, \tag{A9}$$

where ϕ_0 is defined in Eq. (4). The expected utility under the perturbed strategy is

$$E[u(\widetilde{W}_1^T+z\widetilde{W}_1^D+W_0\phi_0R_{f1}(z_{B1}-z_{S1})\delta^D)]. \tag{A10}$$

Since $R_{f_s1}>R_{f1}$, it is immediate to verify that $z_{B1}>z_{S1}$, and therefore the perturbation $(\alpha_0^T+\delta^T, \alpha_0^D-\delta^D)$ with $\delta^D>0$ and δ^T defined in Eq. (A9) leads to an increase in wealth without adding any risk. Since preferences are non-satiated, we can continue to perturb as far as the borrowing and short-selling constraints do not bind. This means that one of the following two cases will prevail: (i) $\alpha_0^{T*}\in[0, 1]$, $\alpha_0^{D*}=0$, or (ii) $\alpha_0^{T*}=1$, $\alpha_0^{D*}\in[0, 1]$. □

Proof of Proposition 1. The single-account solution $\bar{\alpha}_1^{\mathrm{T}}$ in the taxable account-only problem (see, for example, Ingersoll, 1987, p. 243) is

$$\alpha_{1M}^{\mathrm{T}} = \frac{1}{2}R_{f1}\left(\frac{1}{R_{f1}-R_{L1}}-\frac{1}{R_{H1}-R_{f1}}\right). \tag{A11}$$

Similarly, the solution $\bar{\alpha}_1^{\mathrm{D}}$ in the tax-deferred account-only problem is

$$\alpha_{1M}^{\mathrm{D}} = \frac{1}{2}r_{f1}\left(\frac{1}{r_{f1}-r_{L1}}-\frac{1}{r_{H1}-r_{f1}}\right). \tag{A12}$$

The solution in Eq. (A11) refers to the case in which $\phi_1=0$, while Eq. (A12) refers to the case in which $\phi_1=1$. Using the replication argument introduced in Lemma 1 and adapting it to the case of $z_1=1$, it is possible to show that bonds are preferred in the tax-deferred account at time $t=1$. This allows us to solve the problem in a neighborhood of $\phi_1=0$ by searching for a solution in the taxable account while assuming $\alpha_1^{\mathrm{D}}=0$. This implies solving a simpler single-account problem whose solution is

$$\alpha_1^{\mathrm{T}*} = \alpha_{1M}^{\mathrm{T}}\left(1+\frac{\phi_1}{1-\phi_1}\frac{r_{f1}}{R_{f1}}\right). \tag{A13}$$

Notice that the above solution is increasing in ϕ_1 because we assume $\alpha_{1M}^{\mathrm{T}}\in[0, 1]$. Since stocks in the taxable account are always preferred to stocks in the tax-deferred account, Eq. (A13) will be the solution for all the values of ϕ_1 for which $\alpha_1^{\mathrm{T}*}\leq 1$. Imposing this condition yields a unique size ϕ_1^{A} of the tax-deferred account:

$$\phi_1^{\mathrm{A}} = \frac{\frac{R_{H1}}{R_{H1}-R_{f1}}-\frac{R_{L1}}{R_{f1}-R_{L1}}}{\frac{R_{H1}}{R_{H1}-R_{f1}}-\frac{R_{L1}}{R_{f1}-R_{L1}}-r_{f1}\left(\frac{1}{R_{H1}-R_{f1}}-\frac{1}{R_{f1}-R_{L1}}\right)} \tag{A14}$$

such that the portfolio in the taxable account is given by expression (A13) for $\phi\leq\phi_1^{\mathrm{A}}$ and equal to one for $\phi>\phi_1^{\mathrm{A}}$.

To obtain the solution for ϕ_1 close to 1, we can perturb the solution of the tax-deferred account-only problem in Eq. (A12), assuming that the optimal stock holding in the taxable account is $\alpha_1^{\mathrm{T}}=1$. Before doing so, we need to ensure that, for large values of ϕ_1, stocks are always preferred in the taxable account. This is easily shown by relying on a variation of Lemma 1 in which the reduced-form objective function takes the form

$$J_0'(W_0^{\mathrm{T}}, W_0^{\mathrm{D}}) = \max_{\{\alpha_0^{\mathrm{T}},\alpha_0^{\mathrm{D}}\}} E_0[\log(C_1)+\log(z_1'\widetilde{W}_1^{\mathrm{T}}+\widetilde{W}_1^{\mathrm{D}})] \tag{A15}$$

with $z_1'>0$ and repeating the analysis from the point of view of the tax-deferred account. This allows us to establish that investors are always better off preferring stocks in the taxable account. We can then obtain the solution for ϕ_1 close to 1 by perturbing the solution of the tax-deferred account-only problem in Eq. (A12), while assuming that the optimal stock holding in the taxable account is $\alpha_1^{\mathrm{T}}=1$. The solution for this single-account problem is

$$\alpha_1^{\mathrm{D}*} = \alpha_{1M}^{\mathrm{D}} + \frac{(1-\phi_1)}{2\phi_1}\left(\frac{R_{L1}}{r_{f1}-r_{L1}}-\frac{R_{H1}}{r_{H1}-r_{f1}}\right)$$

$$= \alpha_{1M}^{D} + \frac{(1-\phi_1)}{2\phi_1}\left(\left(\frac{R_{L1}-R_{f_s1}}{r_{f1}-r_{L1}} - \frac{R_{H1}-R_{f_s1}}{r_{H1}-r_{f1}}\right) + R_{f_s1}\left(\frac{1}{r_{f1}-r_{L1}} - \frac{1}{r_{H1}-r_{f1}}\right)\right) \tag{A16}$$

$$= \alpha_{1M}^{D}\left(1 + \frac{1-\phi_1}{\phi_1}\frac{R_{f_s1}}{r_{f1}}\right) - \frac{1-\phi_1}{\phi_1}(1-\tau_{S1}), \tag{A17}$$

where the last equality follows by the definition of R_{f_s1} in Eq. (3). It is possible to show that, since $0<\alpha_{1M}^{D}<\alpha_{1M}^{T}<1$, the quantity $\frac{(1-\phi_1)}{2\phi_1}\left(\frac{R_{L1}}{r_{f1}-r_{L1}} - \frac{R_{H1}}{r_{H1}-r_{f1}}\right)$ in Eq. (A16) is increasing in ϕ_1. Hence, to be feasible, the solution must satisfy the constraint $\alpha_1^{D*} \geq 0$. Imposing such a constraint on Eq. (A17) yields the cutoff level,

$$\phi_1^{B} = \frac{\frac{R_{H1}}{r_{H1}-r_{f1}} - \frac{R_{L1}}{r_{f1}-r_{L1}}}{\frac{R_{H1}}{r_{H1}-r_{f1}} - \frac{R_{L1}}{r_{f1}-r_{L1}} - r_{f1}\left(\frac{1}{r_{H1}-r_{L1}} - \frac{1}{r_{f1}-r_{L1}}\right)}, \tag{A18}$$

such that the portfolio in the tax-deferred account is given by expression (A17) for $\phi \geq \phi_1^{B}$ and is equal to zero otherwise. Notice that $\phi_1^{A}<\phi_1^{B}$. □

Proof of Corollary 1. Given the optimal portfolios (α_1^{T*}, α_1^{D*}) in Proposition 1, we can derive the indirect utility by substituting the optimal portfolio into the objective function (5) at time $t=1$.

$$J_1(W_1^{T}, W_1^{D}) = \begin{cases} \log(A_1) + \log(\widehat{W}_1^{T}) & \text{if } 0 \leq \phi_1 \leq \phi_1^{A} \\ \log(B_1) + \frac{1}{2}\log\left(W_1^{T} + \frac{r_{f1}}{R_{H1}}W_1^{D}\right) + \frac{1}{2}\log\left(W_1^{T} + \frac{r_{f1}}{R_{L1}}W_1^{D}\right) & \text{if } \phi_1^{A} < \phi_1 \leq \phi_1^{B}, \\ \log(C_1) + \log(\widehat{W}_1^{D}) & \text{if } \phi_1^{B} < \phi_1 \leq 1 \end{cases} \tag{A19}$$

where $\widehat{W}_1^{T} = W_1^{T} + W_1^{D}\frac{r_{f1}}{R_{f1}}$, $\widehat{W}_1^{D} = W_1^{T}\frac{R_{f_s1}}{r_{f1}} + W_1^{D}$, and

$$A_1 = \frac{R_{f1}(R_{H1}-R_{L1})}{2\sqrt{(R_{H1}-R_{f1})(R_{f1}-R_{L1})}} \tag{A20}$$

$$B_1 = \sqrt{R_{H1}R_{L1}} \tag{A21}$$

$$C_1 = \frac{R_{H1}(r_{f1}-r_{L1}) + R_{L1}(r_{H1}-r_{f1})}{2\sqrt{(r_{H1}-r_{f1})(r_{f1}-r_{L1})}} \cdot \frac{r_{f1}}{R_{f_s1}}. \tag{A22}$$

By Definition 1, the relevant expression of the indirect utility used to define the tax subsidy is the one for which $\phi_1=0$. This justifies the second equality in Eq. (13). The expression for the subsidy in Eq. (14) follows directly from Eqs. (A19) and (13). □

Proof of Lemma 2. By direct substitution, it is possible to verify that any portfolio (α_0^{T}, α_0^{D}) such that

$$\alpha_0^{\mathrm{D}} = \frac{r_{f0}(1-\tau_{\mathrm{S0}})\alpha_0^{\mathrm{T}}}{R_{f0} + (r_{f0}-1)(\tau_{\mathrm{B0}}-\tau_{\mathrm{S0}})\alpha_0^{\mathrm{T}}} \tag{A23}$$

yields

$$\phi_{H1} = \phi_{L1} = \frac{\phi_0 r_{f0}}{\phi_0 r_{f0} + (1-\phi_0)(R_{f0} + \alpha_0^{\mathrm{T}}(r_{f0}-1)(\tau_{\mathrm{B0}}-\tau_{\mathrm{S0}}))}. \quad \square \tag{A24}$$

Proof of Proposition 2. The proof is by construction. The objective function at time 0 is given by the indirect utility (A19) in Proposition 1. Given a current value of ϕ_0 and a portfolio (α_0^{T}, α_0^{D}), we denote by ϕ_{H1}, ϕ_{L1} the future realizations of $\tilde{\phi}_1$, where,

$$\tilde{\phi}_1 = \frac{\phi_0(\alpha_0^{\mathrm{D}}(\tilde{r}_0-r_{f0}) + r_{f0})}{(1-\phi_0)(\alpha_0^{\mathrm{T}}(\tilde{R}_0-R_{f0}) + R_{f0}) + \phi_0(\alpha_0^{\mathrm{D}}(\tilde{r}_0-r_{f0}) + r_{f0})}. \tag{A25}$$

Clearly, there are 9 possible cases, depending on how $\tilde{\phi}_1$ relates to ϕ_1^{A} and ϕ_1^{B} in Eqs. (A14) and (A18), respectively. Since the objective function is concave, the first order conditions (FOCs) are necessary and sufficient for a maximum. To construct the solution, for each of the possible 9 regions discussed below, we solve for the portfolio satisfying the FOCs and then determine the values of $\tilde{\phi}_0$ for which the obtained portfolio indeed produces ϕ_{H1} ϕ_{L1} belonging to the region of interest.

1. *Region AA*. $0 \le \phi_{H1} < \phi_1^{\mathrm{A}}$, $0 \le \phi_{L1} < \phi_1^{\mathrm{A}}$. If a portfolio ($\alpha_0^{\mathrm{T}}$, α_0^{D}) leads to this region, from Eq. (A19), the objective function takes the form

$$J_1(W_1^{\mathrm{T}}, W_1^{\mathrm{D}}) = \log(A_1) + \frac{1}{2}\log(\widehat{W}_{H1}^{\mathrm{T}}) + \frac{1}{2}\log(\widehat{W}_{L1}^{\mathrm{T}}) \tag{A26}$$

where $\widehat{W}_{H1}^{\mathrm{T}} = W_{H1}^{\mathrm{T}} + W_{H1}^{\mathrm{D}}\frac{r_{f1}}{R_{f1}}$, $\widehat{W}_{L1}^{\mathrm{T}} = W_{L1}^{\mathrm{T}} + W_{L1}^{\mathrm{D}}\frac{r_{f1}}{R_{f1}}$ with W_{H1}^{T}, W_{L1}^{T} defined in Eq. (6) and W_{H1}^{D}, W_{L1}^{D} defined in Eq. (7). The FOCs with respect to α_0^{T} and α_0^{D} are

$$\frac{R_{H0}-R_{f0}}{\widehat{W}_{H1}^{\mathrm{T}}} + \frac{R_{L0}-R_{f0}}{\widehat{W}_{L1}^{\mathrm{T}}} = 0 \tag{A27}$$

$$\frac{r_{H0}-r_{f0}}{\widehat{W}_{H1}^{\mathrm{T}}} + \frac{r_{L0}-r_{f0}}{\widehat{W}_{L1}^{\mathrm{T}}} = 0. \tag{A28}$$

Since $\tau_{\mathrm{S0}}=\tau_{\mathrm{B0}}=\tau$, $R_{H0}-R_{f0}=(1-\tau)(r_{H0}-r_{f0})$, $R_{L0}-R_{f0}=(1-\tau)(r_{L0}-r_{f0})$ and the two FOCs reduce to the following condition

$$\widehat{W}_{L1}^{\mathrm{T}}(r_{H0}-r_{f0}) + \widehat{W}_{H1}^{\mathrm{T}}(r_{L0}-r_{f0}) = 0. \tag{A29}$$

The subspace of portfolios satisfying the above equation is

$$\alpha_0^{\mathrm{T}*} = \frac{(\phi_0 r_{f0} r_{f1} - \phi_0 r_{f0} R_{f1} + r_{f0} R_{f1})(-2r_{f0} + r_{H0} + r_{L0})}{2(1-\phi_0) R_{f1}(1-\tau)(r_{H0}-r_{f0})(r_{f0}-r_{L0})} - \alpha_0^{\mathrm{D}*} \frac{\phi_0 r_{f1}}{(1-\phi_0) R_{f1}(1-\tau)}. \quad \text{(A30)}$$

After some manipulations, the above set of solutions can be expressed as

$$\alpha_0^{\mathrm{T}*} = \alpha_{0M}^{\mathrm{T}} - \frac{\phi_0 r_{f1}}{(1-\phi_0)(1-\tau) R_{f1}} \delta_{\mathrm{A}} \quad \text{(A31)}$$

$$\alpha_0^{\mathrm{D}*} = \alpha_{0M}^{\mathrm{D}} + \delta_{\mathrm{A}}, \quad \delta_{\mathrm{A}} \in \Delta(\phi_0) \cap \Delta_{\mathrm{A}}(\phi_0), \quad \text{(A32)}$$

where $\Delta(\phi_0)$ is the set of all possible δ_{A} such that the portfolio $(\alpha_0^{\mathrm{T}*}, \alpha_0^{\mathrm{D}*})$ in Eqs. (A31)–(A32) satisfies the borrowing and short-selling constraints for a given ϕ_0,

$$\Delta(\phi_0) \equiv \{\delta : \alpha_0^{\mathrm{T}*}(\phi_0, \delta_{\mathrm{A}}) \in [0,1], \quad \alpha_0^{\mathrm{D}*}(\phi_0, \delta_{\mathrm{A}}) \in [0,1]\}, \quad \text{(A33)}$$

and $\Delta_{\mathrm{A}}(\phi_0) \subset \Delta(\phi_0)$ is the set of δ_{A} such that $\phi_{H1}, \phi_{L1} < \phi_1^{\mathrm{A}}$,

$$\Delta_{\mathrm{A}}(\phi_0) \equiv \{\delta_{\mathrm{A}} : \phi_{H1}(\alpha_0^{\mathrm{T}*}(\phi_0, \delta_{\mathrm{A}}), \alpha_0^{\mathrm{D}*}(\phi_0, \delta_{\mathrm{A}})) \leq \phi_1^{\mathrm{A}}, \quad \phi_{L1}(\alpha_0^{\mathrm{T}*}(\phi_0, \delta_{\mathrm{A}}), \alpha_0^{\mathrm{D}*}(\phi_0, \delta_{\mathrm{A}})) \leq \phi_1^{\mathrm{A}}\}. \quad \text{(A34)}$$

The solution belongs to the region AA and satisfies the constraints if and only if $\delta_{\mathrm{A}} \in \Delta_{\mathrm{A}}(\phi_0) \cap \Delta(\phi_0)$. Imposing these conditions, we arrive at the following equivalent characterizations of the above set:

$$\delta_{\mathrm{A}} \in \Delta_{\mathrm{A}}(\phi_0) \cap \Delta(\phi_0) \Leftrightarrow \delta_{\mathrm{A}} \in \{\delta : \delta \in [\delta_{\mathrm{A}}^{-}(\phi_0), \delta_{\mathrm{A}}^{+}(\phi_0)]\} \quad \text{(A35)}$$

where

$$\delta_{\mathrm{A}}^{-}(\phi_0) = \max\left[-\bar{\Delta}_{\mathrm{A}}(\phi_0), -\alpha_{0M}^{\mathrm{D}}, -(1-\alpha_{0M}^{\mathrm{T}}) \frac{(1-\phi_0)(1-\tau) R_{f1}}{\phi_0 r_{f1}}\right], \quad \text{(A36)}$$

$$\delta_{\mathrm{A}}^{+}(\phi_0) = \min\left[\bar{\Delta}_{\mathrm{A}}(\phi_0), 1-\alpha_{0M}^{\mathrm{D}}, \alpha_{0M}^{\mathrm{T}} \frac{(1-\phi_0)(1-\tau) R_{f1}}{\phi_0 r_{f1}}\right], \quad \text{(A37)}$$

and

$$\bar{\Delta}_{\mathrm{A}}(\phi_0) = -\frac{R_{f1}(r_{H0}-r_{L0})(R_{f0}\phi_1^{\mathrm{A}} + \phi_0(r_{f0}(1-\phi_1^{\mathrm{A}})))}{2\phi_0(r_{H0}-r_{f0})(r_{f0}-r_{L0})(r_{f1}(1-\phi_1^{\mathrm{A}}) + r_{f1})\phi_1^{\mathrm{A}}}. \quad \text{(A38)}$$

The first element in the max and min operators above guarantees that $\delta_{\mathrm{A}} \in \Delta_{\mathrm{A}}(\phi_0)$. The second and third elements of the max and min operators guarantee that $\delta_{\mathrm{A}} \in \Delta(\phi_0)$ for which portfolio constraints are satisfied. It is possible to verify that the highest possible value of ϕ_0 such that $\Delta_{\mathrm{A}}(\phi_0) \cap \Delta(\phi_0) \neq \emptyset$, is equal to

$$\phi_0^{\mathrm{A}} = \frac{\phi_1^{\mathrm{A}}}{\phi_1^{\mathrm{A}} + (1-\phi_1^{\mathrm{A}})\frac{r_{f0}}{R_{f0}}}. \quad \text{(A39)}$$

Moreover, note that $\bar{\Delta}_{\mathrm{A}}(\phi_0)>0$ for $\phi_0 \le \phi_0^{\mathrm{A}}$. This can be seen by observing that $\bar{\Delta}_{\mathrm{A}}(\phi_0^{\mathrm{A}})=0$ and that $\bar{\Delta}_{\mathrm{A}}(\phi_0)$ is a decreasing function of ϕ_0. Hence, if $\alpha_{0M}^{\mathrm{T}}, \alpha_{0M}^{\mathrm{D}} \in [0, 1]$, the arguments in the max(·) are negative and the arguments of the min(·) are positive, proving that $\delta_{\mathrm{A}}^{-}<0<\delta_{\mathrm{A}}^{+}$.

2. *Region AB*. $0 \le \phi_{H1} < \phi_1^{\mathrm{A}}$, $\phi_1^{\mathrm{A}} \le \phi_{L1} < \phi_1^{\mathrm{B}}$. The objective function takes the form

$$J_1(W_1^{\mathrm{T}}, W_1^{\mathrm{D}}) = \frac{1}{2}(\log(A_1) + \log(B_1)) + \frac{1}{2}\log(\widehat{W}_{H1}^{\mathrm{T}}) + \frac{1}{4}\log(W_{LH1}) + \frac{1}{4}\log(W_{LL1}), \tag{A40}$$

where $W_{LH1} \equiv W_{L1}^{\mathrm{T}} + \frac{r_{f1}}{R_{H1}} W_{L1}^{\mathrm{D}}$ and $W_{LH1} \equiv W_{L1}^{\mathrm{T}} + \frac{r_{f1}}{R_{L1}} W_{L1}^{\mathrm{D}}$. The FOCs with respect to α_0^{T} and α_0^{D} are

$$2\frac{R_{H0}-R_{f0}}{\widehat{W}_{H1}^{\mathrm{T}}} + \frac{R_{L0}-R_{f0}}{W_{LH1}} + \frac{R_{L0}-R_{f0}}{W_{LL1}} = 0 \tag{A41}$$

$$2\frac{r_{f1}}{R_{f1}}\frac{r_{H0}-r_{f0}}{\widehat{W}_{H1}^{\mathrm{T}}} + \frac{r_{f1}}{R_{L1}}\frac{r_{L0}-r_{f0}}{W_{LH1}} + \frac{r_{f1}}{R_{H1}}\frac{r_{L0}-r_{f0}}{W_{LL1}} = 0, \tag{A42}$$

with the unique solution

$$\begin{aligned}\alpha_0^{\mathrm{T}*} = {} & \frac{1}{4(1-\phi_0)(1-\tau)(r_{f0}-r_{H0})(R_{f1}-R_{H1})(r_{f0}-r_{L0})(R_{f1}-R_{L1})} \cdot \\ & [r_{f0}(4(\phi_0-1)R_{f0}(R_{f1}-R_{H1})(R_{f1}-R_{L1}) + \phi_0 r_{f1}(r_{H0}-r_{L0})(2R_{f1}-R_{H1}-R_{L1})) \\ & + (\phi_0-1)R_{f0}(-2(r_{H0}+r_{L0}){R_{f1}}^2 + (3r_{H0}+r_{L0})(R_{H1} \\ & + R_{L1})R_{f1} - 4r_{H0}R_{H1}R_{L1})]\end{aligned} \tag{A43}$$

$$\begin{aligned}\alpha_0^{\mathrm{D}*} = {} & \frac{1}{4\phi_0 r_{f1}(r_{f0}-r_{H0})(R_{f1}-R_{H1})(r_{f0}-r_{L0})(R_{f1}-R_{L1})} \cdot \\ & [R_{f0}R_{f1}(r_{H0}-r_{L0})(R_{f1}(r_{H1}+R_{L1}) - 2r_{H1}R_{L1}) + \phi_0(r_{f0}r_{f1}(-4r_{H0}R_{f,1}^2 \\ & + (3r_{H0}+r_{L0})(r_{H1}+R_{L1})R_{f1} + 4r_{f0}(R_{f1}-r_{H1})(R_{f1}-R_{L1}) - 2r_{H1}(r_{H0} \\ & + r_{L0})R_{L1}) - R_{f0}R_{f1}(r_{H0}-r_{L0})(R_{f1}(r_{H1}+R_{L1}) - 2r_{H1}R_{L1})))]\end{aligned} \tag{A44}$$

Direct substitution of the above solution into Eq. (A25) shows that $\phi_{L1}=\phi_1^{\mathrm{A}}$ and that $\phi_{H1} \le \phi_1^{\mathrm{A}}$ for $\phi_0 \le \phi_0^{\mathrm{A}}$, with ϕ_0^{A} defined in Eq. (A62). Hence, the region AB is simply a subset of the region AA, for which $\phi_{L1}=\phi_1^{\mathrm{A}}$, and we can ignore it in the solution.

3. *Region AC*. $0 \le \phi_{H1} < \phi_1^{\mathrm{A}}$, $\phi_1^{\mathrm{B}} \le \phi_{L1} < 1$. The objective function takes the form

$$J_1(W_1^{\mathrm{T}}, W_1^{\mathrm{D}}) = \frac{1}{2}(\log(A_1) + \log(C_1)) + \frac{1}{2}\log(\widehat{W}_{H1}^{\mathrm{T}}) + \frac{1}{2}\log(\widehat{W}_{L1}^{\mathrm{D}}), \tag{A45}$$

where $\widehat{W}_{L1}^{\mathrm{D}} = W_{L1}^{\mathrm{T}}\frac{R_{f_s1}}{r_{f1}} + W_{L1}^{\mathrm{D}}$. The FOCs with respect to α_0^{T} and α_0^{D} are

$$\frac{R_{H0}-R_{f0}}{\widehat{W}_{H1}^{\mathrm{T}}} + \frac{R_{fs1}}{r_{f1}}\frac{R_{L0}-R_{f0}}{\widehat{W}_{L1}^{\mathrm{D}}} = 0 \tag{A46}$$

$$\frac{r_{f1}}{R_{f1}}\frac{r_{H0}-r_{f0}}{\widehat{W}^{\mathrm{T}}_{H1}}+\frac{r_{L0}-r_{f0}}{\widehat{W}^{\mathrm{D}}_{L1}}=0, \tag{A47}$$

which do not admit solutions. Hence, region AC is not viable.

4. *Region BA*. $\phi_1^{\mathrm{A}}\le\phi_{H1}<\phi_1^{\mathrm{B}}$, $0\le\phi_{L1}<\phi_1^{\mathrm{A}}$. The argument is similar to the one provided for region AB in point 2 above. In this case, it is possible to show that $\phi_{H1}=\phi_1^{\mathrm{A}}$ and that $\phi_{L1}\le\phi_1^{\mathrm{A}}$ for $\phi_0\le\phi_0^{\mathrm{A}}$. Hence, the region BA is simply a subset of the region AA, for which $\phi_{H1}=\phi_1^{\mathrm{A}}$, and we can ignore it in the solution.

5. *Region BB*. $\phi_1^{\mathrm{A}}\le\phi_{H1}<\phi_1^{\mathrm{B}}$, $\phi_1^{\mathrm{A}}\le\phi_{L1}<\phi_1^{\mathrm{B}}$. The objective function is

$$J_1(W_1^{\mathrm{T}},W_1^{\mathrm{D}})=\log(B_1)+\frac{1}{4}(\log(W_{LL1})+\log(W_{LH1})+\log(W_{HL1})+\log(W_{HH1})), \tag{A48}$$

where $W_{LL1}\equiv W_{L1}^{\mathrm{T}}+\frac{r_{f1}}{R_{L1}}W_{L1}^{\mathrm{D}}$, $W_{LH1}\equiv W_{L1}^{\mathrm{T}}+\frac{r_{f1}}{R_{L1}}W_{L1}^{\mathrm{D}}$, $W_{HL1}\equiv W_{H1}^{\mathrm{T}}+\frac{r_{f1}}{R_{L1}}W_{H1}^{\mathrm{D}}$ and $W_{HH1}\equiv W_{H1}^{\mathrm{T}}+\frac{r_{f1}}{R_{H1}}W_{H1}^{\mathrm{D}}$. The FOCs are

$$\frac{R_{L0}-R_{f0}}{W_{LL1}}+\frac{R_{L0}-R_{f0}}{W_{LH1}}+\frac{R_{H0}-R_{f0}}{W_{HL1}}+\frac{R_{H0}-R_{f0}}{W_{HH1}}=0$$

$$\frac{\frac{r_{f1}}{R_{L1}}(r_{L0}-r_{f0})}{W_{LL1}}+\frac{\frac{r_{f1}}{R_{H1}}(r_{L0}-r_{f0})}{W_{LH1}}+\frac{\frac{r_{f1}}{R_{L1}}(r_{H0}-r_{f0})}{W_{HL1}}+\frac{\frac{r_{f1}}{R_{H1}}(r_{H0}-r_{f0})}{W_{HH1}}=0$$

which, using $\tau_{\mathrm{S0}}=\tau_{\mathrm{B0}}=\tau$, are equivalent to

$$\frac{(r_{L0}-r_{f0})W_{HL1}+(r_{H0}-r_{f0})W_{LL1}}{W_{HL1}W_{LL1}}+\frac{(r_{L0}-r_{f0})W_{LH1}+(r_{H0}-r_{f0})W_{HH1}}{W_{LH1}W_{HH1}}=0$$

$$\frac{r_{f1}}{R_{L1}}\frac{(r_{L0}-r_{f0})W_{HL1}+(r_{H0}-r_{f0})W_{LL1}}{W_{HL1}W_{LL1}}+\frac{r_{f1}}{R_{H1}}\frac{(r_{L0}-r_{f0})W_{LH1}+(r_{H0}-r_{f0})W_{HH1}}{W_{LH1}W_{HH1}}=0.$$

Since wealth is positive, the above system of equation is satisfied for $(\alpha_0^{\mathrm{T}},\alpha_0^{\mathrm{D}})$ solving

$$(r_{L0}-r_{f0})W_{HL1}+(r_{H0}-r_{f0})W_{LL1}=0 \tag{A49}$$

$$(r_{L0}-r_{f0})W_{LH1}+(r_{H0}-r_{f0})W_{HH1}=0. \tag{A50}$$

The unique solution to the above equations is

$$\alpha_0^{\mathrm{T}*}=\frac{R_{f0}}{2}\left[\frac{1}{R_{f0}-R_{L0}}-\frac{1}{R_{H0}-R_{f0}}\right]\equiv\alpha_{0M}^{\mathrm{T}} \tag{A51}$$

$$\alpha_0^{\mathrm{D}*}=\frac{r_{f0}}{2}\left[\frac{1}{r_{f0}-r_{L0}}-\frac{1}{r_{H0}-r_{f0}}\right]\equiv\alpha_{0M}^{\mathrm{D}}. \tag{A52}$$

Using the definition of ϕ_1 in Eq. (A25), it is possible to show that the ϕ_{H1}, $\phi_{L1}\in[\phi_1^{\mathrm{A}},\phi_1^{\mathrm{B}}]$ under the above solution if and only if $\phi_0>\phi_0^{\mathrm{A}}$ and $\phi_0<\phi_0^{\mathrm{B}}$, where ϕ_0^{A} is given in Eq. (A62) and

$$\phi_0^{\mathrm{B}}=\frac{\phi_1^{\mathrm{B}}}{(1-\phi_1^{\mathrm{B}})\frac{r_{f0}}{R_{f0}}+\phi_1^{\mathrm{B}}}. \tag{A53}$$

6. *Region BC*. $\phi_1^{\rm A} \le \phi_{H1} < \phi_1^{\rm B}$, $\phi_1^{\rm B} \le \phi_{L1} < 1$. The argument is similar to the one provided for region AB in point 2 above. In this case, it is possible to show that $\phi_{H1} = \phi_1^{\rm B}$ and that $\phi_{L1} \ge \phi_1^{\rm B}$ for $\phi_0 \le \phi_0^{\rm B}$. Hence, the region BC is simply a subset of the region CC (below), for which $\phi_{H1} = \phi_1^{\rm B}$, and we can ignore it in the solution.

7. *Region CA*. $\phi_1^{\rm B} \le \phi_{H1} < 1$, $0 \le \phi_{L1} < \phi_1^{\rm A}$. As in the proof for region CA in point 3 above, it is possible to show that there are no solutions in this region.

8. *Region CB*. $\phi_1^{\rm B} \le \phi_{H1} < 1$, $\phi_1^{\rm A} \le \phi_{L1} < 1$. The argument is similar to the one provided for region AB in point 2 above. In this case, it is possible to show that $\phi_{L1} = \phi_1^{\rm B}$ and that $\phi_{H1} \ge \phi_1^{\rm B}$ for $\phi_0 \le \phi_0^{\rm B}$. Hence, the region BC is simply a subset of the region CC (below), for which $\phi_{L1} = \phi_1^{\rm B}$, and we can ignore it in the solution.

9. *Region CC*. $\phi_1^{\rm B} \le \phi_{H1} < 1$, $\phi_1^{\rm B} \le \phi_{L1} < 1$. The proof follows the same logic as the argument used for region AA in point 1 above. The objective function is

$$J_1(W_1^{\rm T}, W_1^{\rm D}) = \log(C_1) + \frac{1}{2}\log(\widehat{W}_{H1}^{\rm D}) + \frac{1}{2}\log(\widehat{W}_{L1}^{\rm D}). \tag{A54}$$

Solving the FOCs, we obtain the following subspace of solutions:

$$\alpha_0^{\rm T*} = \alpha_{0M}^{\rm T} - \frac{\phi_0 r_{f1}}{(1-\phi_0)(1-\tau)R_{f_s1}}\delta_{\rm C} \tag{A55}$$

$$\alpha_0^{\rm D*} = \alpha_{0M}^{\rm D} + \delta_{\rm C}, \quad \delta_{\rm A} \in \Delta(\phi_0) \cap \Delta_{\rm C}(\phi_0), \tag{A56}$$

where, as before, $\Delta(\phi_0)$ is the set of all possible $\delta_{\rm C}$ such that the portfolios ($\alpha_0^{\rm T*}$, $\alpha_0^{\rm D*}$) in Eqs. (A55)–(A56) satisfy the borrowing and short-selling constraints for a given ϕ_0, and $\Delta_{\rm C}(\phi_0) \subset \Delta(\phi_0)$ is the set of $\delta_{\rm C}$ such that $\phi_{H1}, \phi_{L1} > \phi_1^{\rm B}$,

$$\Delta_{\rm C}(\phi_0) \equiv \{\delta : \phi_{H1}(\alpha_0^{\rm T*}(\phi_0\delta), \alpha_0^{\rm D*}(\phi_0\delta)) \ge \phi_1^{\rm B}, \quad \phi_{L1}(\alpha_0^{\rm T*}(\phi_0\delta), \alpha_0^{\rm D*}(\phi_0\delta)) \ge \phi_1^{\rm B}\}. \tag{A57}$$

The solution belongs to the region CC and satisfies the constraints if and only if $\delta_{\rm A} \in \Delta_{\rm C}(\phi_0) \cap \Delta(\phi_0)$. Imposing these conditions, we arrive at the following equivalent characterizations of the above set:

$$\delta_{\rm C} \in \Delta_{\rm C}(\phi_0) \cap \Delta(\phi_0) \Leftrightarrow \delta_{\rm C} \in \{\delta : \delta \in [\delta_{\rm C}^-(\phi_0), \delta_{\rm C}^+(\phi_0)]\}, \tag{A58}$$

where,

$$\delta_{\rm C}^-(\phi_0) = \max\left[-\overline{\Delta}_{\rm C}(\phi_0), -\alpha_{0M}^{\rm D}, -(1-\alpha_{0M}^{\rm T})\frac{(1-\phi_0)(1-\tau)R_{f_s1}}{\phi_0 r_{f1}}\right], \tag{A59}$$

$$\delta_{\rm C}^+(\phi_0) = \min\left[\overline{\Delta}_{\rm C}(\phi_0), 1-\alpha_{0M}^{\rm D}, \alpha_{0M}^{\rm T}\frac{(1-\phi_0)(1-\tau)R_{f_s1}}{\phi_0 r_{f1}}\right], \tag{A60}$$

and

$$\overline{\Delta}_{\rm C}(\phi_0) = -\frac{R_{f_s1}(r_{H0}-r_{L0})(R_{f0}\phi_1^{\rm B} - \phi_0(R_{f0} + (r_{f0}-1)\tau(1-\phi_1^{\rm B})))}{2\phi_0(r_{H0}-r_{f0})(r_{f0}-r_{L0})(R_{f_s1}(1-\phi_1^{\rm B}) + r_{f1}\phi_1^{\rm B})}. \tag{A61}$$

The first element in the max and min operators above guarantees that $\delta_{\rm C} \in \Delta_{\rm C}(\phi_0)$. The second and third elements of the max and min operators guarantee that $\delta_{\rm A} \in \Delta(\phi_0)$ for which portfolio constraints are satisfied. It is possible to verify that the lowest possible value of ϕ_0 such that $\Delta_C(\phi_0) \cap \Delta(\phi_0) \neq \emptyset$, is equal to

$$\phi_0^B = \frac{\phi_1^B}{(1-\phi_1^B)\frac{r_{f0}}{R_{f0}} + \phi_1^B}. \tag{A62}$$

Moreover, note that $\bar{\Delta}_C(\phi_0) > 0$ for $\phi_0 \leq \phi_0^A$. This can be seen by observing that $\bar{\Delta}_C(\phi_0^B) = 0$ and that $\bar{\Delta}_C(\phi_0)$ is an increasing function of ϕ_0. Hence, if $\alpha_{0M}^T, \alpha_{0M}^D \in [0, 1]$, the arguments in the max(·) are negative and the arguments in the min(·) are positive, proving that $\delta_C^- < 0 < \delta_C^+$. □

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