To Recognize or Not to Recognize Assets when Future Benefits are Uncertain

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1. Introduction:

When should an expenditure of resources be recognized as an asset and when should it be treated as a current period expense? This is a key question in accounting measurement. FASB’s Statement of Financial Accounting Concepts No. 6 states that expenditures should be recognized as assets if they give rise to probable future benefits that are controllable by the entity. In most instances there is considerable uncertainty in the amount of such future benefits and FASB recognizes this fact by incorporating the word “probable” in its definition. But “probable” is too vague for such an important distinction. Can the notion of “probable” be refined by explicitly identifying a critical level of uncertainty beyond which assets should not be recognized? What factors determine this critical level of uncertainty? Is there a risk-return tradeoff in determining an appropriate threshold? In this paper we provide a perspective that sheds light on these important questions.

Answers to the above questions must presuppose some fundamental objective that is served by distinguishing between assets and operating expenses. FASB believes that separating assets and expenses results in a “better” measure of the firm’s performance. But, it is unclear what “better” means. In the academic literature, it is often argued that the performance measure is better because it facilitates more precise predictions of future cash flows by disentangling persistent and transitory components of past cash flows. Although this argument is intuitively appealing, it is incomplete because it is based on the assumption that there is some fixed stochastic process that generates these future cash flows. What is missing is the recognition that firms consciously choose a variety of actions that affect the distribution of their future cash flows. For example, firms choose how much to invest, in what projects to invest, what assets to acquire, what changes to make in their operating decisions, how to manage risk,
etc. and all these choices have significant effects on future cash flows. Kanodia and Sapra [2016] argue that in making these choices, firms are concerned with “price performance” in the capital market rather, than their own assessment of future cash flows. Therefore, the information that is reflected in capital market prices has a powerful effect on what firms choose to do. The critical question becomes: Are prices in the capital market sufficiently informed to sustain economically efficient decisions by firms? Following Stein [1989], Kanodia and Mukherji [KM, 1996] used such a real effects perspective to show that accounting measurements and reports on a firm’s investments have a powerful effect on the firm’s choice of investment. Specifically, the failure to separate investment from operating expenses produces corporate myopia and significant under-investment by firms. Such a real effects perspective provides a compelling rationale for measuring and reporting on a firm’s investment, and we use it to study the issues addressed here.

Kanodia, Sapra and Venugopalan [KSV 2004] extended the Kanodia and Mukherji [1996] analysis to study the effect of measurement imprecision on the question of whether a firm’s intangible investments should be measured. KSV study this issue in a setting where the firm chooses a mix of tangible and intangible investments. They showed that if tangibles are measured precisely but the measurement of intangibles is noisy and error prone, there is a relevance-reliability tradeoff that determines whether intangible investments should be measured and reported to the capital market. KSV [2004] derived the result that intangibles should be measured only if the measurement is sufficiently precise and the effect of intangible investments on future cash flows is sufficiently large. In all other cases it is better to leave intangible investments comingled with operating expenses.
In both KM [1996] and KSV[2004], the profitability of the firm’s investment is common knowledge among managers and the market. The only unknown that the market needs to assess is the amount of the firm’s investment. Kanodia and Lee [1998] studied a firm’s investment in settings where managers are privately informed about the profitability of investment, and Kanodia, Singh and Spero [KSS, 2005] studied settings where managers are privately informed about both the profitability and the amount of the firm’s investment. In such settings, the capital market seeks to extract information about the profitability of investment from accounting measurements of the amount of investment. This gives rise to Spence-type [1974] signaling equilibria or noisy signaling equilibria in which firms’ over-invest rather than under-invest. KSS [2005] show that, in the presence of such signaling incentives, perfect measurement of the firm’s investment is generally undesirable and noise in accounting measurement results in capital market prices that provide better incentives for investment. KSS[2005] also establish that, in some settings, there exists an interior level of noise in accounting measurement that sustains first best levels of investment.

The present paper builds upon the KSS [2005] framework, in that both the amount of the firm’s investment and the profitability of investment are privately known to the firm’s managers. But, we seek answers to a different kind of question than that studied in KSS [2005]. The focus of the current paper is not on measurement noise regarding the amount of investment, but on the fundamental uncertainty that is inherent in virtually every investment. This is the uncertainty in future benefits/returns and this uncertainty is independent of measurement noise. Casual observation suggests that if these future benefits are highly uncertain, accountants are reluctant to capitalize the expenditure as an investment and would rather expense it. But the critical threshold of uncertainty must surely be tempered by the size of the mean of the distribution of future benefits. The higher the expected future benefit from investment, the
greater the tolerance for the uncertainty in future benefits. While such accounting treatments seem intuitive, we don’t fully understand why they are rational accounting responses. In order to investigate these questions we incorporate several new features into the economic setting that are not present in KSS [2005].

Distinct from KSS [2005], we assume that the firm is concerned not just with the immediate price response of the capital market, but with both current and future capital market prices and that future market prices are much better informed than the current market price. Consequently, the under-investment incentive due to non-measurement/non-recognition of the firm’s investment, while still present, is not as severe as in previous models. Also, distinct from KSS [2005], when the firm’s investment is not measured, the observed net cash flow of the firm simultaneously provides information about the firm’s operating profits from assets in place as well as the amount of its new investment. We show that in such settings, even when the firm’s investment is not measured and reported, there is still a signaling aspect to the firm’s choice of investment. Given that non-recognition of the firm’s investment triggers both under-investment and over-investment incentives, we show that the firm under-invests at some levels of profitability and over-invests at other levels of profitability. But, when the firm’s investment is perfectly measured and reported, only the signaling role remains and the signaling incentive is much stronger than is the case where the firm’s investment is not measured. Consequently, if the accounting regime is designed ex ante before the firm learns its private information, the optimal choice of accounting regime could tilt towards either non-recognition or recognition of the firm’s investment depending upon the severity and direction of distortions in investment at various levels of profitability. We show that both the level of uncertainty in future benefits as well as the level of expected future benefits are key factors in determining which accounting regime is optimal.
2. The Economic Setting

We consider a firm in a 3 date economy, dates zero, one and two. At date zero the firm already has assets in place that generate operating profits $\tilde{x}_1$ and $\tilde{x}_2$ at dates 1 and 2, respectively. We assume that $\tilde{x}_1$ is distributed Normal with mean $\mu$ and variance $\sigma^2_x$ and $\tilde{x}_2 = gx_1 + \tilde{e}$, where $g > 0$ is a publicly known persistence parameter and $\tilde{e}$ is distributed Normal with mean 0 and variance $\sigma^2_e$. This specification implies that operating profits from assets in place have positive covariance, with $\text{Cov}(\tilde{x}_1, \tilde{x}_2) = g\sigma^2_x > 0$ and the unconditional expectation of $\tilde{x}_2$ is $g\mu$. At date zero, the firm chooses some amount $k$ to invest in a new project whose returns are realized at date two and are independent of the returns to the assets in place. The date two return from the new investment is strictly concave in the amount of investment and equals $(1 + \theta)k - 0.5k^2$. The parameter $\theta$ represents the profitability of the new project and is drawn from a Normal distribution with mean $\bar{\theta}$ and variance $\sigma^2_{\theta}$. The cash flows of the firm at dates one and two, net of investment expenditures, are:

$$\tilde{z}_1 = \tilde{x}_1 - k,$$

and

$$\tilde{z}_2 = \tilde{x}_2 + (1 + \theta)k - 0.5k^2$$

The firm chooses the amount of new investment, $k$, to maximize its expectation of a weighted average of the firm’s capital market prices at dates one and two. Let $P_1, P_2$ be these capital market prices. The firm’s objective function is: 

1. See Kanodia and Sapra (2016) for a detailed justification of such price based objective functions.
where \( 0 < \alpha < 1 \) and \( E_0(.) \) denotes expectation conditional on the information available to the firm at date zero. We assume that the firm observes the profitability parameter \( \theta \) at date zero, before choosing its investment.

We assume that in all accounting regimes the firm’s net cash flows \( z_1 \) and \( z_2 \) are perfectly observed and communicated to the capital market by the firm’s accounting system as and when these cash flows occur. If, additionally, the capital market knows all that the manager knows at date zero, i.e. if the market also observes \( \theta \) and observes the firm’s choice of \( k \), then we have a first best economy. For expository purposes, it is useful to characterize optimal investment for such a first best setting. Assuming that the firm pays no dividends, the prices in the capital market at dates 1 and 2, in this first best economy, would be:

\[
P_1 = z_1 + E(z_2 | k, \theta) = x_1 - k + g x_1 + (1+\theta)k - 0.5k^2, \tag{1}
\]

\[
P_2 = z_1 + z_2 \tag{2}
\]

Note that the market makes no assessments at date 2, because there are no future cash flows. The date 2 price in the capital market is simply the observed cash accumulation in the firm at that date. Thus, from the firm’s perspective:

\[
E_0(\tilde{P}_1 | k, \theta) = E_0(\tilde{P}_2 | k, \theta) = (1+g)\mu + \theta k - \frac{1}{2}k^2 \tag{3}
\]

Given such price behavior, the first best investment schedule of the firm is: \( k^*(\theta) = \theta \).
In the remainder of the paper, we assume that the capital market does not directly observe $\theta$, nor does it directly observe how much the firm has invested. The firm’s investment is measured, precisely or imprecisely, by the accounting system and reported to outside agents. But the profitability parameter $\theta$ cannot be measured, and even though the manager knows $\theta$ he cannot communicate it by way of cheap talk because there is no mechanism to verify the manager’s claims. We consider three measurement regimes: In the first regime, the firm’s investment is perfectly measured and reported. In the second regime, the accounting system makes no attempt to measure the firm’s investment and merely reports the firm’s net cash flow $z_1$ at date one. Finally, in the third regime the accounting system does measure the firm’s investment but makes random errors in doing so. For this last regime, we consider all variations in the precision of accounting measurement.

Since the date 2 price does not depend upon any inference about future cash flows, $E_0(\tilde{P}_2 | k, \theta)$ is the same in all three accounting regimes and is described by (2) and (3). However,

$$E_0(\tilde{P}_1 | k, \theta) = E_0 \left( z_1 + E[g\tilde{x}_1 + (1 + \tilde{\theta})\tilde{k} - 0.5\tilde{k}^2 | \text{the market's information}] | k, \theta \right)$$  (4)

Equation (4) indicates that the firm’s date zero expectation of the price in the capital market at date 1, is the firm’s date 0 expectation of the market’s date 1 expectation of future cash flows. The law of iterated expectations does not apply because all of the firm’s information at date 0 is not available to the market at date 1. Specifically, the market views $\tilde{k}$ and $\tilde{\theta}$ as random variables and must assess their distribution conditional on the information that the market receives, while the firm knows the exact values of $k$ and $\theta$. 

3. Equilibrium Investment in a Perfect Measurement Regime

With perfect measurement of the firm’s investment, the firm’s choice of \( k \) is revealed to the capital market. But, the market still does not know the profitability of investment \( \theta \) and must make inferences about it in order to rationally price the firm. As in KSS [2005], perfect measurement of investment leads to perfect inferences of \( \theta \) in a Spence-type signaling equilibrium. In such signaling equilibria, the firm is not consciously trying to communicate \( \theta \) through its investment choice. It merely reacts to how its choices are priced in the capital market. However, agents in the capital market are conscious of the fact that the firm chooses its investment after observing \( \theta \) and therefore seek to infer the value of \( \theta \) from the firm’s chosen investment. We characterize such a fully revealing signaling equilibrium below.

Since \( x_i = z_i + k \), perfect knowledge of the firm’s investment also perfectly reveals the date 1 operating profits from assets in place. For any observed \( k \), let \( \hat{\theta}(k) \) be the inferred value of \( \theta \). Then the date 1 price in the capital market is:

\[
P_1 = x_i + gx_i - k + [1 + \hat{\theta}(k)]k - 0.5k^2 \tag{5}
\]

and,

\[
E_0(\tilde{P}_1 | k, \theta) = (1 + g)\mu + \hat{\theta}(k)k - 0.5k^2 \tag{6}
\]

and, as always,

\[
P_2 = z_i + z_2 = x_i + x_2 - k + [1 + \theta]k - 0.5k^2 \tag{7}
\]

and,

\[
E_0(\tilde{P}_2 | k, \theta) = (1 + g)\mu + \theta k - 0.5k^2 \tag{8}
\]
Note that $P_2$ reflects the actual realized value of $\theta$, while $P_1$ reflects the inferred value of $\theta$.

Given such price behavior, the firm’s objective function,

$$\max_k \left[ \alpha E_0(\hat{P}_1 | k, \theta) + (1-\alpha)E_0(\hat{P}_2 | k, \theta) \right]$$

is equivalent to:

$$\max_k \left[ (1+g)\mu + \alpha(\hat{\theta}(k)k - 0.5k^2) + (1-\alpha)(\theta k - 0.5k^2) \right]$$

Therefore the firm’s optimal investment satisfies:

$$\alpha \left[ \hat{\theta}(k) + k \frac{\partial \hat{\theta}}{\partial k} - k \right] + (1-\alpha)(\theta - k) = 0 \quad \text{(9)}$$

Since, in a fully revealing signaling equilibrium, $\hat{\theta}(k(\theta)) = \theta$, equation (9) indicates that the date 1 price in the capital market is overly sensitive to the firm’s investment. This is because investment not only affects future cash flows, it also affects inferences about the profitability of investment. If higher levels of investment lead to higher inferred values of $\theta$, i.e., $\frac{\partial \hat{\theta}}{\partial k} > 0$, the firm over-invests at every $\theta$. We use the Kanodia and Lee [1998] mechanism design approach to characterize a fully revealing signaling equilibrium.

If the firm’s equilibrium investment schedule $k(\theta)$ is fully revealing, an observed investment of $k(\hat{\theta})$ must lead to the inference that, with probability 1, the profitability parameter is $\hat{\theta}$, regardless of what the true profitability parameter is. Since any equilibrium investment schedule must be incentive compatible, a fully revealing $k(\theta)$ must satisfy the incentive compatibility conditions:

$$\theta k(\theta) - 0.5k(\theta)^2 \geq [\alpha \hat{\theta} + (1-\alpha)\theta]k(\hat{\theta}) - 0.5k(\hat{\theta})^2, \forall(\theta, \hat{\theta}) \quad \text{(10)}$$
Analysis of these incentive compatibility conditions yields:

**Proposition 1:**

*In a regime where the accounting system perfectly measures and reports the firm’s investment, the firm’s equilibrium investment schedule is* \( k_p(\theta) = (1 + \alpha)\theta \)

**Proof:** See the Appendix.

Comparing \( k_p(\theta) \) to the first best investment schedule \( k^*(\theta) = \theta \), it is clear that perfect measurement causes the firm to over-invest at every \( \theta \). The intuition for the above result is best seen via a slightly less general mode of analysis that will later be used to also analyze the other two measurement regimes mentioned above. Suppose the market conjectures that the equilibrium investment schedule is linear in \( \theta \), say

\[
k(\theta) = a + b\theta
\]

for some parameters \( \{a, b\} \), that are to be determined. Then, having observed some amount of investment \( k \) the market must infer:

\[
\hat{\theta}(k) = \frac{k - a}{b}
\]

Such inferences imply that the firms’ choice of investment is a solution to:

\[
\text{Max}_{\hat{\theta}} \alpha \left( \frac{k - a}{b} \right) k - 0.5k^2 + (1 - \alpha) \left( \theta k - 0.5k^2 \right),
\]

\( ^2 \text{When } \theta < 0, k_p(\theta) < k^*(\theta) < 0. \text{ Mathematically, } k_p(\theta) < k^*(\theta) \text{ indicates under-investment. But, economically, a greater negative investment when the appropriate level of investment is negative, is like a greater positive investment when the appropriate level of investment is positive. It is useful to interpret both cases as over-investment.} \)
which yields the first order condition:

\[
\alpha \left( \frac{2k}{b} - \frac{a}{b} - k \right) + (1 - \alpha) (\theta - k) = 0,
\]

or, equivalently,

\[
k \left[ 1 - \left( \frac{\alpha}{1 - \alpha} \right) \left( \frac{2b - k}{b} \right) \right] = \theta - \left( \frac{\alpha}{1 - \alpha} \right) \frac{a}{b},
\]

which confirms that \( k(\theta) \) is a linear function of \( \theta \). Matching coefficients with the conjectured investment schedule gives:

\[
b = \left[ 1 - \left( \frac{\alpha}{1 - \alpha} \right) \left( \frac{2b}{b} \right) \right]^{-1}, \text{ and}
\]

\[
a = - \left( \frac{\alpha}{1 - \alpha} \right) \left( \frac{a}{b} \right) b
\]

Solving gives: \( b = 1 + \alpha \) and \( a = 0 \), as in Proposition 1.

**Equilibrium Investment In a No Measurement Regime**

Now, suppose the accounting regime makes no attempt to measure and report on the firm’s investment. In such a regime, the only information that is disclosed to the capital market is the firm’s net cash flows \( z_1 \) at date 1 and \( z_2 \) at date 2. Such lack of information has no affect on the date 2 price since the date 2 price is simply the net cash accumulation in the firm at that date, i.e., \( P_2 = z_1 + z_2 \). However, the date 1 price is strongly affected by the inferences made
by investors in the capital market. Both the firm’s investment $k$ and the firm’s date 1 operating profits $x_1$ are informative about the firm’s future cash flows, but these two components cannot be disentangled because they are aggregated in the observed date 1 net cash flow, $z_1 = x_1 - k$.

The date 1 price in the capital market depends upon investor assessments of $E(k \mid z_1)$, $E(x_1 \mid z_1)$, and $E(\theta \mid z_1)$. We characterize these assessments below.

Since the firm’s choice of investment is independent of $x_1$, in principle, the market’s assessments of $x_1$ should independent of its assessment of $k$. But, given the observation of net cash flow $z_1 = x_1 - k$, any assessment of $x_1$ is also simultaneously an assessment of $k$. The two assessments are no longer independent. In fact, $E(k \mid z_1) = E(x_1 \mid z_1) - z_1$. Additionally, inferences about the quantity of the firm’s investment $k$ lead to inferences about the profitability of investment $\theta$, as argued below. As in the perfect measurement regime, we conjecture, and later confirm, that the equilibrium investment schedule of the firm has the linear form $k(\theta) = a + b\theta$, but the values of the parameters \{a, b\} could differ from their values in the perfect measurement regime. Since, in equilibrium, investors in the capital market know the firm’s investment policy, they must assess:

$$E(\theta \mid z_1) = \frac{E(k \mid z_1) - a}{b}$$  \hspace{1cm} (11)
We now use the above arguments to precisely characterize the market’s assessments. Because $k(\theta) = a + b\theta$ and the prior distribution of $\theta$ is Normal, the prior distribution over the firm’s investment is Normal with $E(k) = a + b\mu = \bar{K}$ and $\text{Var}(k) = b^2 \sigma_\theta^2$. Also,

$$E(x_i | z_i) = \mu + \frac{\text{Cov}(x_i, z_i)}{\text{Var}(z_i)}[z_i - E(z_i)]$$

where,

$$\frac{\text{Cov}(x_i, z_i)}{\text{Var}(z_i)} = \frac{\text{Cov}(x_i, x_i - k)}{\text{Var}(x_i) + \text{Var}(k)} = \frac{\sigma_x^2}{\sigma_x^2 + b^2 \sigma_\theta^2} = \beta_z$$

(12)

and, $E(z_i) = \mu - \bar{K}$. Using these facts, gives

$$E(x_i | z_i) = (1 - \beta_z)\mu + \beta_z (z_i + \bar{K})$$

(13)

Therefore,

$$E(k | z_i) = E(x_i | z_i) - z_i = (1 - \beta_z)(\mu - z_i) + \beta_z \bar{K}$$

(14)

Then (11) implies,

$$E(\theta | z_i) = \frac{(1 - \beta_z)(\mu - z_i) + \beta_z \bar{K} - a}{b} = (1 - \beta_z)\left(\frac{\mu - z_i - a}{b}\right) + \beta_z \bar{\theta}$$

(15)

It is instructive to compare the inferences described in (13), (14), and (15) to inferences in the regime with perfect measurement of investment. In the perfect measurement regime a $1$ increase in investment causes the market’s beliefs of investment to also increase by $1$, and the
market’s beliefs of \( \theta \) to increase by \( \frac{1}{1+\alpha} \), while the market’s beliefs of operating profits remains unchanged. But, when investment is not measured, a $1 increase in investment causes a $1 decrease in \( z_i \) which causes the market’s belief of investment to go up by only \( (1 - \beta_z) \) and simultaneously causes the market’s belief of \( x_i \) to decrease by \( \beta_z \). For example, if \( \beta_z = 0.4 \) (in general, \( 0 < \beta_z < 1 \)), a $1 increase in investment causes the market’s belief of investment to go up by only $0.60 and simultaneously causes the market’s belief of operating profits to decrease by $0.40. Both inferences make it more “costly” to invest in the regime where investment is not measured. The market’s beliefs of \( \theta \) changes by \( \frac{1 - \beta_z}{b} = 0.6 \). If \( b > 0 \), as will be the case, higher investment increases the market’s belief of the profitability of investment even though the investment itself is not observed. So, a signaling incentive remains even when investment is not measured. However, the above analysis suggests that the sensitivity of investment to variations in \( \theta \) will be muted relative to the case where investment is perfectly measured. (We later derive the result that the equilibrium value of \( b \) in this regime satisfies \( 1 < b < 1 + \alpha \) for all parameter values, thus confirming the above intuition).

We proceed to characterize equilibrium prices and equilibrium investment in this regime. We construct equilibrium prices under the conjecture that \( k(\theta) = a + b\theta \) and later verify this conjecture.

\[
P_i(z_i) = z_i + E \left[ x_2 + (1 + \theta)\tilde{k} - 0.5\tilde{k}^2 \mid z_i \right] \\
= z_i + gE(x_i \mid z_i) + E(\tilde{k} \mid z_i) + E(\tilde{\theta}\tilde{k} \mid z_i) - 0.5E(\tilde{k}^2 \mid z_i)
\] (16)
But, since under the conjectured investment schedule \( \theta = \frac{k-a}{b} \),

\[
E(\theta k \mid z_i) = E\left[\left(\frac{k-a}{b}\right)k \mid z_i\right] = \frac{1}{b}E(k^2 \mid z_i) - \frac{a}{b}E(k \mid z_i), \quad \text{and}
\]

\[
E(k^2 \mid z_i) = E^2(k \mid z_i) + Var(k \mid z_i).
\]

Also, since \( k = x_i - z_i \),

\[
Var(k \mid z_i) = Var(x_i - z_i \mid z_i) = Var(x_i \mid z_i) = (1 - \beta_z)\sigma_x^2
\]

Since \( Var(k \mid z_i) \) is a constant that does not depend upon any choice that the firm makes, we will ignore it in subsequent calculations. Inserting the above assessments into (16) and using (13) gives:

\[
P_1(z_i) = z_i + g[(1 - \beta_z)\mu + \beta_z(z_i + \bar{K})] + \left[1 - \frac{a}{b}\right]\left[(1 - \beta_z)(\mu - z_i) + \beta_z\bar{K}\right]
\]

\[
+ \left(1 - \frac{a}{b} - \frac{1}{2}\right)(1 - \beta_z)(\mu - z_i) + \beta_z\bar{K})^2
\]

(17)

From the firm’s date 0 perspective, the only random variable in (17) is the net cash flow \( z_i \).

For any choice of \( k \) the firm assesses \( E_0(z_i \mid k) = \mu - k \). Therefore, the firm’s date 0 expectation of the date 1 price is:

\[
E_0(\tilde{P}_1 \mid k, \theta) = (1 + g)\mu - k + g\beta_z(-k + \bar{K}) + \left[1 - \frac{a}{b}\right]\left[(1 - \beta_z)k + \beta_z\bar{K}\right]
\]

\[
+ \left(1 - \frac{a}{b} - \frac{1}{2}\right)(1 - \beta_z)(k + \beta_z\bar{K})^2
\]

(18)
Thus, in the regime where the firm’s investment is not measured, the firm’s optimization problem \( \max_k \left[ \alpha E_0(\tilde{P}_1 | k, \theta) + (1-\alpha)E_0(\tilde{P}_2 | k, \theta) \right] \) incorporates \( E_0(\tilde{P}_1 | k, \theta) \) as described in (18) and \( E_0(\tilde{P}_2 | k, \theta) \) as described in (8). The first order condition with respect to \( k \) yields:

\[
\alpha \left[ -1 - g \beta_z + \left( 1 - \frac{a}{b} \right) (1 - \beta_z) + 2 \left( \frac{1}{b} - \frac{1}{2} \right) [(1 - \beta_z)k + \beta_z \bar{K}] (1 - \beta_z) \right] + (1-\alpha)(\theta - k) = 0
\]

which can be expressed as:

\[
k \left[ 1 + \frac{\alpha - 2b}{1-\alpha} (1 - \beta_z)^2 \right] = \theta + \left( \frac{\alpha}{1-\alpha} \right) \left[ -\beta_z (1 + g) - \frac{a}{b} (1 - \beta_z) + \left( \frac{2 - b}{b} \right) (1 - \beta_z) \beta_z \bar{K} \right]
\]

Thus, the investment schedule implied by capital market prices does indeed have the linear form \( k(\theta) = a + b \theta \) that we conjectured. Matching coefficients gives:

\[
b = \left[ 1 - \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{2-b}{b} \right) (1 - \beta_z)^2 \right]^{-1}, \quad (19)
\]

and,

\[
a = \left( \frac{\alpha}{1-\alpha} \right) \left[ -\beta_z (1 + g) - \frac{a}{b} (1 - \beta_z) + \left( \frac{2-b}{b} \right) (1 - \beta_z) \beta_z \bar{K} \right] b \quad (20)
\]

With some algebraic manipulation of (19) and collecting terms in \( b \), gives,
The exogenous parameters that affect the firm’s investment are $\sigma^2_x$ which describes the uncertainty in operating profits from assets in place, $\sigma^2_\theta$ which describes the uncertainty in future benefits from a unit of new investment, $\bar{\theta}$ which describes the expected benefit from a unit of new investment, the weight on the first period price $\alpha$, and $g$ which describes the persistence in operating profits from assets in place. From (21) it is obvious that the coefficient $b$ in the firm’s investment schedule is a function only of $\sigma^2_\theta, \sigma^2_x, \alpha$, and is independent of the parameters $\bar{\theta}$ and $g$. There is no guarantee that equilibrium value of $b$ is unique for all parameter values. Below, we establish the properties of $b$ that are critical to our results.

**Proposition 2**

*All equilibrium values of $b$ must lie in the interval $1 < b < 1 + \alpha$ regardless of parameter values, There always exists at least one equilibrium value of $b$ in this interval.*

Recall that the equilibrium in the perfect measurement of investment regime has $b = 1 + \alpha$ and the first best investment schedule has $b = 1$. Proposition 2 indicates that the non-measurement of investment makes the firm’s investment less sensitive to the profitability of investment relative to the case where investment is perfectly measured. But, relative to first best, investment is still overly sensitive to the profitability of investment. These results are due to
to the fact that when the firm’s investment is not measured, the firm’s net cash flow
simultaneously conveys information about the operating profits from assets in place and the
profitability of new investment. A lower net cash flow signals lower operating profits from
assets in place (i.e. lower value of \( x_i \)) and higher new investment and, therefore, a higher
profitability of new investment (i.e., a higher value of \( \theta \)). Since the firm’s net cash flow plays
a dual informational role, its effect on each of the two inferences is muted.

**Proposition 3:**

The coefficient \( b \) is strictly increasing in \( \sigma_\theta^2 \), strictly decreasing in \( \sigma_s^2 \) and
\( b \to (1 + \alpha) \) as \( \sigma_\theta^2 \to \infty \)

*Proof: See the Appendix.*

The result that \( 1 < b < 1 + \alpha \) does not imply that the firm under-invests at all values of
\( \theta \) relative to the case of perfect measurement, nor does it imply over-investment relative to first
best, since nothing is yet known about the constant \( a \) in the firm’s investment schedule
\( k(\theta) = a + b\theta \). The constant \( a \) augments investment at every \( \theta \) if \( a > 0 \) and decreases
investment at every \( \theta \) if \( a < 0 \). The sign of \( a \) is therefore important in determining whether
investment should be measured or not measured. Below, we establish sufficient conditions on
exogenous parameter values under which \( a \) is positive or negative.

Substituting \( \bar{K} = a + b\bar{\theta} \) in (20) and collecting terms involving \( a \), yields:

\[
a \left[ 1 + \left( \frac{\alpha}{1 - \alpha} \right) (1 - \beta_z) [1 - (2 - b)\beta_z] \right] = \left( \frac{\alpha}{1 - \alpha} \right) b \beta_z \left[ -(-1 + g) + (2 - b)(1 - \beta_z)\bar{\theta} \right]
\]
Since $b$ and $\beta$ are independent of $\bar{\theta}$, it is apparent from (22) that $a$ is a linear function of $\bar{\theta}$ and can be expressed as $a(\bar{\theta}) = -m_0 + m_1 \bar{\theta}$, where

$$m_0 = \frac{\left(\frac{\alpha}{1-\alpha}\right) b \beta_z (1 + g)}{1 + \left(\frac{\alpha}{1-\alpha}\right) (1 - \beta_z) \left[1 - (2-b)\beta_z\right]}$$

and

$$m_1 = \frac{\left(\frac{\alpha}{1-\alpha}\right) b \beta_z (2-b) (1 - \beta_z)}{1 + \left(\frac{\alpha}{1-\alpha}\right) (1 - \beta_z) \left[1 - (2-b)\beta_z\right]}.$$

Therefore $a(\bar{\theta}) = 0$ at $\bar{\theta} = \frac{m_0}{m_1} = \frac{1+g}{(2-b)(1-\beta_z)} > 0$. Additionally,

$$a(\bar{\theta}) > 0 \text{ at } \bar{\theta} > \frac{m_0}{m_1}, \text{ and } a(\bar{\theta}) < 0 \text{ at } \bar{\theta} < \frac{m_0}{m_1}.$$

Unfortunately, $\frac{m_0}{m_1}$ cannot be expressed in terms of exogenous parameters. Working with the upper and lower bounds on $b$ that we identified in Proposition 2, we show:
Proposition 4

(i) If $\sigma_\theta^2 > \sigma_x^2$ and $\bar{\theta} < \left(1 + \frac{\sigma_x^2}{\sigma_\theta^2}\right)(1 + g)$ then, necessarily, $a < 0$

(ii) If $\sigma_\theta^2 > \sigma_x^2$, and $\bar{\theta} > \left(1 + \frac{1}{(1+\alpha)^2}\right)\left(\frac{1+g}{(1-\alpha)}\right)$ then, necessarily, $a > 0$

Proof: See the Appendix.

Recognition versus Non Recognition of the Firm’ Investment

We now compare the “perfect measurement” regime to the “no measurement” regime to gain insights into the question of whether the assets acquired via the firm’s investment should be measured and recognized or not measured and not recognized. We are particularly interested in examining how the answer depends on $\bar{\theta}$ which describes the expected return per unit of investment and on $\sigma_\theta^2$ which captures the uncertainty in returns. We also wish to investigate the possibility of a risk-return tradeoff in determining whether assets should or should not be recognized.

We begin by developing a useful metric for comparison of the two regimes. Ex post, the social value $w$ produced by the firm is simply the sum of its cash flows over the two periods of the economy, i.e., $w = z_1 + z_2$. For given $\theta$ and a given investment schedule $k(\theta)$ this quantity is $w(\theta) = x_1 + x_2 + \theta k(\theta) - 0.5k^2(\theta)$. What is different across measurement regimes is the amount of the firm’s investment at each $\theta$. Using $k^*(\theta) = \theta$, $w(\theta)$ can be expressed as:
\(w(\theta) = x_1 + x_2 + 0.5\theta^2 - 0.5\left[k(\theta) - k^*(\theta)\right]^2\)

Since the first three terms in the above expression are common to all regimes, we can compare regimes solely in terms of the expected social loss due to deviations of investment from first best:

\[\Omega = E_\theta \left[k(\theta) - k^*(\theta)\right]^2\]

We have shown that in all of the regimes that we have examined, the equilibrium investment schedule of the firm is described by the linear form \(k(\theta) = a + b\theta\). Using this fact, and using \(k^*(\theta) = \theta\) gives:

\[\Omega(a,b) = E\left[(a+b\bar{\theta}) - \bar{\theta}\right]^2 = E\left[a + (b-1)\bar{\theta}\right]^2\]

and, taking the expectation:

\[\Omega(a,b) = \left[a + (b-1)\bar{\theta}\right]^2 + (b-1)^2\sigma_\theta^2\]  \(25\)

Differences in investment across regimes can be described by differences in the equilibrium values of the parameters \(\{a, b\}\). Henceforth, we use the notation \(\{a_p, b_p\}\) to denote the equilibrium parameter values in the regime with perfect measurement of investment, the notation \(\{a_N, b_N\}\) for the “no measurement” regime and the notation \(\{a^*, b^*\}\) for the first best economy. We have established, so far, that \(a^* = a_p = 0\), \(b^* = 1\), and \(b_p = 1 + \alpha\). We have also established that, depending upon parameter values, \(a_N\) could be positive, negative or zero and that \(1 < b_N < 1 + \alpha\).
From (25), the expected social loss in the perfect measurement regime is:

$$\Omega(a_p, b_p) = \alpha^2 \bar{\theta}^2 + \alpha^2 \sigma_{\bar{\theta}}^2$$

Now, consider the Non Measurement regime. We know that \(b_N - 1 < b_p - 1 = \alpha\) for all parameter values, so \((b_N - 1)^2 \sigma_{\bar{\theta}}^2 < \alpha^2 \sigma_{\bar{\theta}}^2\) for all parameter values. Therefore,

$$\left(a_N + (b_N - 1)\bar{\theta}\right)^2 \leq \alpha^2 \bar{\theta}^2 \quad (26)$$

is sufficient to guarantee that the expected social loss from non-recognition is strictly less than the expected social loss from measurement and recognition. Since \(b_N - 1 < \alpha\), it follows immediately that if \(a_N = 0\), i.e., \(\bar{\theta} = \frac{m_0}{m_1}\) which is true at some \(\bar{\theta}_0 > 0\), non-recognition is superior to recognition. Also at \(\bar{\theta} = 0\), \(a_N < 0\), so (26) is violated which means that at \(\bar{\theta} = 0\), non-recognition is not guaranteed to be superior to recognition. However,

$$a_N < 0, \quad a_N + (b_N - 1)\bar{\theta} > 0 \quad (27)$$

is a sufficient set of conditions to guarantee that non-recognition of the asset is the preferred choice. Thus, there is some interval of strictly positive \(\bar{\theta}\) values at which non-recognition is guaranteed to be superior to recognition. Working with (27), we characterize such an interval in terms of upper and lower bounds.

Proposition 4 established sufficient conditions under which \(a_N < 0\). We proceed to establish sufficient conditions under which \(a_N + (b_N - 1)\bar{\theta} > 0\) that are consistent with the conditions under which \(a_N < 0\).
**Proposition 5**

Non-recognition of the firm’s investment is better than recognition if:

(i) \( \sigma_\theta^2 > \sigma_x^2 \) and,

\[ (i) \quad \sigma_\theta^2 > \sigma_x^2 \quad \text{and}, \quad (28) \]

(ii) \( \frac{\alpha}{4 - 3\alpha} < \frac{\sigma_x^2}{\sigma_\theta^2} \) and,

\[ (ii) \quad \frac{\alpha}{4 - 3\alpha} < \frac{\sigma_x^2}{\sigma_\theta^2} \quad \text{and}, \quad (29) \]

(iii) \( \left( \frac{4 - 2\alpha}{4 - 3\alpha} \right)(1 + g) < \tilde{\theta} < \left( 1 + \frac{\sigma_x^2}{\sigma_\theta^2} \right)(1 + g) \)

\[ (iii) \quad \left( \frac{4 - 2\alpha}{4 - 3\alpha} \right)(1 + g) < \tilde{\theta} < \left( 1 + \frac{\sigma_x^2}{\sigma_\theta^2} \right)(1 + g) \quad (30) \]

Proof: See the Appendix.

Condition (ii) of Proposition 5 merely insures that \( \frac{4 - 2\alpha}{4 - 3\alpha} < \left( 1 + \frac{\sigma_x^2}{\sigma_\theta^2} \right) \) so that the interval of \( \tilde{\theta} \) identified in (iii) is non-empty. It is a relatively mild condition that says that the firm’s objective function should not assign too much weight to the first period price. For example if \( \sigma_x^2 = 0.6\sigma_\theta^2 \) then condition (ii) requires that \( \alpha < 0.85 \), and if \( \sigma_x^2 = 0.2\sigma_\theta^2 \) then (ii) requires that \( \alpha < 0.5 \).

We know ask whether there are values of \( \tilde{\theta} \) where measurement and recognition is better than non-recognition of the firm’s investment. Proposition 5 suggests that this is likely to be the case if the parameters \{\( \alpha, \sigma_\theta^2, \sigma_x^2 \)\} satisfy conditions (28) and (29) and \( \tilde{\theta} \) is sufficiently large. Formally, recognition is better than non-recognition if:

\[ \left[ a_N + (b_N - 1)\tilde{\theta} \right]^2 > \left[ \alpha\tilde{\theta} \right]^2 + \left[ \alpha^2 - (b_N - 1)^2 \right] \sigma_\theta^2 \]
Since \( \left[ \alpha^2 - (b_N - 1)^2 \right] \sigma_\theta^2 \) is a positive constant, in the sense that it is independent of \( \bar{\theta} \), recognition is better if \( a_N > 0, \bar{\theta} > 0 \) and \( a_N + (b_N - 1)\bar{\theta} \) is sufficiently bigger than \( \alpha\bar{\theta} \).

Substituting \( a_N = -m_0 + m_1 \bar{\theta} \), the above requirement is equivalent to:

\[
-m_0 + (m_1 + b_N - 1)\bar{\theta} \gg \alpha\bar{\theta}
\]

where \( \gg \) denotes “sufficiently bigger.” Clearly, the above inequality will hold if

\( (m_1 + b_N - 1) > \alpha \) and \( \bar{\theta} \) is large enough. Now, \( m_1 + b_n - 1 > 0 \) and is a function solely of \( \{\alpha, \sigma_\theta^2, \sigma_x^2\} \). We will identify conditions on these parameters that insure that

\( (m_1 + b_N - 1) > \alpha \) so that recognition is better than non-recognition if the expected benefit \( \bar{\theta} \) is big enough. [Authors note: This work remains to be done]

**Variations in the Uncertainty of Benefits:**

We now hold the expected return \( \bar{\theta} \) fixed and examine whether non-recognition dominates recognition if the uncertainty parameter \( \sigma_\theta^2 \) crosses a critical threshold. We have, so far, done the analysis only for the case where \( \bar{\theta} \) is fixed at zero. At \( \bar{\theta} = 0 \), \( a_N < 0 \) and the social loss due to non-recognition is \( a_N^2 + (b_N - 1)^2 \sigma_\theta^2 \) whereas the social loss due to recognition is \( \alpha^2 \sigma_\theta^2 \). Therefore, non-recognition is better if and only if:

\[
a_N^2 + (b_N - 1)^2 \sigma_\theta^2 < \alpha^2 \sigma_\theta^2
\]

Substituting the expressions for \( a_N \) and \( b_N \) from (22) and (21) the above inequality reduces to
\[ \sigma^2_\theta > \frac{a^2}{\alpha^2 - (b-1)^2} = \frac{(1 + g)^2 \alpha \beta_z^2 \left[ 1 - \alpha (2 - \beta_z) \beta_z \right]^2 \left[ 1 + \alpha (1 - 2 (2 - \beta_z) \beta_z) \right]^2}{(1 - \alpha) \left[ 1 - \alpha (2 - \beta_z)^2 \beta_z^2 \right] \left[ 1 - \alpha \beta_z \left( 4 - 2 \beta_z - \alpha (1 + (1 - \beta_z) \beta_z) \right) \right]^2} \]  

(31)

(This inequality was obtained after considerable algebraic manipulation done in Mathematica).

We verified with the help of Mathematica that the RHS of the above inequality is strictly increasing in \( \beta_z \). Since \( \beta_z \equiv \frac{\sigma^2_z}{\sigma^2_x + b^2 \sigma^2_\theta} \) is strictly decreasing in \( \sigma^2_\theta \), the right hand side of (31) is strictly decreasing in \( \sigma^2_\theta \). Therefore there exists a unique threshold \( \sigma^2_\theta \), beyond which non-recognition is better than recognition. We have shown:

**Proposition 6**

*With \( \bar{\theta} \) fixed at \( \bar{\theta} = 0 \), non-recognition is better than recognition if the uncertainty parameter \( \sigma^2_\theta \) is sufficiently large.*

To be continued ………….
Appendix

**Proof of Proposition 1**

The proof of Proposition 1 uses lemma 1 below.

**Lemma 1**

*Given the pricing rules described in (5) and (6), an investment schedule* \( k(\theta) \) *is incentive compatible if and only if:*

(i) \( V'(\theta) = (1 - \alpha)k(\theta), \) *and*

(ii) \( k(\theta) \) *is increasing.*

We first prove necessity. Given an investment schedule \( k(\theta), \) let

\[
V(\theta) \equiv \theta k(\theta) - 0.5k^2(\theta), \forall \theta
\]  

(A1)

Then, the incentive compatibility requirements, described in (6), can be stated as:

\[
V(\theta) \geq V(\hat{\theta}) - (1 - \alpha)k(\hat{\theta})[\hat{\theta} - \theta]
\]  

(A2)

and,

\[
V(\hat{\theta}) \geq V(\theta) - (1 - \alpha)k(\theta)[\theta - \hat{\theta}],
\]

for all pairs \( \{\theta, \hat{\theta}\} \). The above inequalities are equivalent to:

\[
(1 - \alpha)k(\theta)[\hat{\theta} - \theta] \leq V(\hat{\theta}) - V(\theta) \leq (1 - \alpha)k(\hat{\theta})[\hat{\theta} - \theta]
\]  

(A3)

(A3) implies that if the investment schedule is incentive compatibility then it must be true that \( k(\hat{\theta}) \geq k(\theta), \forall \hat{\theta} > \theta \). This monotone requirement implies that the investment schedule \( k(\theta) \) must be continuous almost everywhere and therefore \( k(\hat{\theta}) \to k(\theta) \) as \( \hat{\theta} \to \theta \). Dividing (A1)
by \((\hat{\theta} - \theta)\) and taking the limit as \(\hat{\theta} \rightarrow \theta\) yields \(V'(\theta) = (1 - \alpha)k(\theta)\) as a necessary condition for incentive compatibility.

To prove sufficiency, we now show that any investment schedule that satisfies conditions (i) and (ii) must be incentive compatible. Consider \(\hat{\theta} > \theta\). From (i),

\[
\int_{\hat{\theta}}^{\theta} V'(t) dt = (1 - \alpha)\int_{\hat{\theta}}^{\theta} k(t) dt \quad \text{and,}
\]

from (ii), \(\int_{\hat{\theta}}^{\theta} k(t) dt \leq \int_{\hat{\theta}}^{\theta} k(\hat{\theta}) dt\). Therefore, conditions (i) and (ii) imply

\[
V(\hat{\theta}) - V(\theta) \leq (1 - \alpha) k(\hat{\theta}) [\hat{\theta} - \theta],
\]

which is equivalent to (A1). Similar reasoning applies when \(\hat{\theta} < \theta\). This completes the proof of the Lemma.

Since any equilibrium investment schedule must be incentive compatible, Lemma 1 implies that any fully revealing signaling equilibrium must have \(k'(\theta) > 0\) and for any \(\hat{\theta} < \theta\),

\[
\int_{\hat{\theta}}^{\theta} V'(t) dt = V(\theta) - V(\hat{\theta}) = (1 - \alpha)\int_{\hat{\theta}}^{\theta} k(t) dt
\]

Then since, by definition, \(V(\theta)\) satisfies (A1), it follows that any equilibrium investment schedule must be such that:

\[
\theta k(\theta) - 0.5k^2(\theta) = (1 - \alpha)\int_{\hat{\theta}}^{\theta} k(t) dt + V(\theta) \tag{A4}
\]

Differentiating (A4) with respect to \(\theta\) implies that \(k(\theta)\) must satisfy:
\[ k(\theta) + \theta k'(\theta) - k(\theta)k'(\theta) = (1 - \alpha)k(\theta). \]

which yields the differential equation:

\[ k'(\theta)[\theta - k(\theta)] = -\alpha k(\theta) \quad (A5) \]

Since \( k'(\theta) > 0 \) is necessary for a fully revealing signaling equilibrium, only those solutions to (A5) that have the property \( k(\theta) > \theta \) if \( k(\theta) > 0 \) and \( k(\theta) < \theta \) if \( k(\theta) < 0 \) are admissible.

It is easily verified that \( k(\theta) = (1 + \alpha)\theta \) is a solution to (A5) and has the desirable properties.

Q.E.D.

**Proof of Proposition 2**

We have shown, in the paper, that the coefficient \( b \) in the firm’s linear investment schedule must satisfy:

\[ b = \left[ 1 - \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{2-b}{b} \right) (1 - \beta_2)^2 \right]^{-1} \quad (A6) \]

Take the inverse of (A6). This yields:

\[ \frac{1}{b} = 1 - \left( \frac{\alpha}{1-\alpha} \right) \left( \frac{2-b}{b} \right) (1 - \beta_2)^2 \]

Multiplying through by \( b \) and rearranging terms, the above equation is equivalent to:

\[ b - 1 = \left( \frac{\alpha}{1-\alpha} \right) (2-b)(1 - \beta_2)^2 \quad (A7) \]
where, \( (1 - \beta_z) \equiv \frac{b^2\sigma_x^2}{\sigma_x^2 + b^2\sigma_\theta^2} \)

We first claim that \((A7)\) cannot be satisfied at any \( b \leq 1 \). The left hand side of \((A7)\) is strictly negative at all \( b < 1 \) and equals 0 at \( b = 1 \). The right hand side of \((A7)\) equals 0 at \( b = 0 \) because \((1 - \beta_z) \to 0\) as \( b \to 0 \). Also, the right hand side of \((A7)\) is strictly positive at all values of \( b \) in the interval \( 0 < b \leq 1 \). Therefore the right hand side of \((A7)\) is strictly greater than the left hand side of \((A7)\) at all \( b \leq 1 \). Next, we claim that there is necessarily a value of \( b \) in the interval \( 1 < b < 1 + \alpha \) at which \((A7)\) is satisfied. We have shown that at \( b = 1 \), the left hand side of \((A7)\) is strictly less than the right hand side. At \( b = (1 + \alpha) \) the left hand side equals \( \alpha \) while the right hand side equals \( \alpha \frac{(1+\alpha)^2\sigma_\theta^2}{\sigma_x^2 + (1+\alpha)^2\sigma_\theta^2} < \alpha \), so the right hand side is less than the left hand side of \((A7)\). Since both sides of \((A7)\) are continuous in \( b \) there must be at least one value of \( b \) satisfying \( 1 < b < 1 + \alpha \) at which \((A7)\) holds. Finally, we claim that \((A7)\) cannot hold at any value of \( b \geq 1 + \alpha \). We have already shown that \( b = 1 + \alpha \) cannot be a solution to \((A7)\). At any \( b > 1 + \alpha \), the left hand side of \((A7)\) is strictly greater than \( \alpha \) and the right hand side is strictly less than \( \left( \frac{\alpha}{1-\alpha} \right)[2-(1+\alpha)](1-\beta_z)^2 < \alpha \).

Q.E.D.

**Proof of Proposition 3**

We have shown, in the paper, that the coefficient \( b \) in the firm’s linear investment schedule must satisfy:
Collecting terms in \( b \) gives, after some algebraic manipulation,

\[
b = \left[ 1 - \left( \frac{\alpha}{1 - \alpha} \right) \left( \frac{2 - b}{b} \right) (1 - \beta)^2 \right]^{-1}
\]

Using the implicit function theorem and taking the derivative of \( b \) with respect to \( \sigma_\theta^2 \) gives:

\[
\frac{\partial b}{\partial \sigma_\theta^2} = -\frac{1}{\left[ 1 - \left( \frac{1}{2 + \left( \frac{1 - \alpha}{\alpha} \right) \left( \frac{1}{1 - \beta} \right)^2 \right) \right]^2} \frac{1}{\left[ 2 + \left( \frac{1 - \alpha}{\alpha} \right) \left( \frac{1}{1 - \beta} \right)^2 \right]^2} \left( \frac{1 - \alpha}{\alpha} \right) \left( \frac{2}{(1 - \beta)^3} \right) \left( \frac{\partial \beta}{\partial \sigma_\theta^2} + \frac{\partial \beta}{\partial \sigma_\theta^2} \frac{\partial b}{\partial \sigma_\theta^2} \right)
\]

Using \( \beta = \frac{\sigma_x^2}{\sigma_x^2 + b^2 \sigma_\theta^2} \), we obtain \( \frac{\partial \beta}{\partial b} = -\frac{2b\sigma_x^2\sigma_\theta^2}{(\sigma_x^2 + b^2 \sigma_\theta^2)^2} \) and \( \frac{\partial \beta}{\partial \sigma_\theta^2} = -\frac{b^3 \sigma_x^2}{(\sigma_x^2 + b^2 \sigma_\theta^2)^2} \)
Substituting these derivatives into the expression for \( \frac{\partial b}{\partial \sigma_\theta^2} \) and collecting terms gives:

\[
\frac{\partial b}{\partial \sigma_\theta^2} = \frac{1}{1 - \frac{2}{1 - \alpha}} \left( 2 \left( 1 - \frac{1 - \alpha}{\alpha} \right) \left( \frac{1}{1 - \beta} \right) \right)^2 \left( \frac{1 - \alpha}{\alpha} \right) \left( \frac{2}{1 - \beta} \right) \frac{b^2 \sigma_x^2}{\sigma_x^2 + b^2 \sigma_\theta^2}
\]

Both the numerator and the denominator in the above expression are strictly positive. Thus

\[
\frac{\partial b}{\partial \sigma_\theta^2} > 0. \quad \text{The limit property of } b \text{ is due to the fact that:}
\]

\[
\frac{1}{1 - \beta_z} = \frac{\sigma_x^2 + b^2 \sigma_\theta^2}{b^2 \sigma_\theta^2} \to 1 \quad \text{as} \quad \sigma_\theta^2 \to \infty, \quad \text{by which (A8) implies} \quad b \to 1 + \alpha. \quad \text{Similarly,}
\]
we compute \( \frac{\partial b}{\partial \sigma_x^2} \) using the implicit function theorem and obtain that

\[
\frac{\partial b}{\partial \sigma_x^2} = \frac{\frac{1}{1} 2 \left( \frac{1-\alpha}{\alpha} \right) \left( \frac{1}{1-\beta} \right)^2 \left( \frac{1-\alpha}{\alpha} \right) 2 (\sigma_x^2 + b^2 \sigma_\theta^2)^2}{1 - \frac{1}{2 + \left( \frac{1-\alpha}{\alpha} \right) \left( \frac{1}{1-\beta} \right)^2} \frac{1}{2 + \left( \frac{1-\alpha}{\alpha} \right) \left( \frac{1}{1-\beta} \right)^2} \left( \frac{1-\alpha}{\alpha} \right) 2 (\sigma_x^2 + b^2 \sigma_\theta^2)^2}
\]

The denominator is the same as that in \( \frac{\partial b}{\partial \sigma_\theta^2} \) and is strictly positive. The numerator is strictly negative. Thus \( \frac{\partial b}{\partial \sigma_x^2} < 0 \), which completes the proof.

**Proof of Proposition 4**

We first establish sufficient conditions for \( a < 0 \). We previously established in (22) that

\[
\alpha \left[ 1 + \left( \frac{\alpha}{1-\alpha} \right) (1-\beta_\zeta) \left[ 1 - (2-b)\beta_\zeta \right] \right] = \left( \frac{\alpha}{1-\alpha} \right) b\beta_\zeta \left[ -1 + g + (2-b)(1-\beta_\zeta)\theta \right]
\]

\[(A9)\]
Since $b > 1$ as proved in Proposition 2, and $0 < \beta_z < 1$, so $1 - (2 - b)\beta_z > 0$. Therefore, the factor multiplying $a$ in the left hand side of (A8) is strictly positive.

It follows that $a < 0$ if and only if $\left[-(1 + g) + (2 - b)(1 - \beta_z)\bar{\theta} \right] < 0$, i.e.,

$a < 0$ if and only if:

$$(2 - b)\left(\frac{b^2\sigma^2_{\theta}}{\sigma^2_x + b^2\sigma^2_{\theta}}\right)\bar{\theta} < (1 + g) \quad (A10)$$

We now show that if $\sigma^2_{\theta} > \sigma^2_x$, the left hand side of (A10) must necessarily be decreasing at all equilibrium values of $b$. It will then follow that if (A10) is satisfied at $b = 1$ then (A10) must be satisfied at all equilibrium values of $b$.

With some simple algebraic manipulation,

$$\frac{\partial}{\partial b} \left\{ (2 - b)\left(\frac{b^2\sigma^2_{\theta}}{\sigma^2_x + b^2\sigma^2_{\theta}}\right) \right\} = \left(\frac{b^2\sigma^2_{\theta}}{\sigma^2_x + b^2\sigma^2_{\theta}}\right)\left[-1 + \left(\frac{2(2-b)}{b}\right)\left(\frac{\sigma^2_x}{\sigma^2_x + b^2\sigma^2_{\theta}}\right)\right]$$

In the above expression, the term $\left(\frac{2(2-b)}{b}\right)\left(\frac{\sigma^2_x}{\sigma^2_x + b^2\sigma^2_{\theta}}\right)$ is strictly decreasing at all values of $b < 2$. Therefore if at $b = 1$, $\left(\frac{2(2-b)}{b}\right)\left(\frac{\sigma^2_x}{\sigma^2_x + b^2\sigma^2_{\theta}}\right) < 1$ then the left hand side of (A9) must be strictly decreasing at all equilibrium values of $b$. At $b = 1$ the above inequality is
\[
\frac{2\sigma_x^2}{\sigma_x^2 + \sigma_{\theta}^2} < 1, \text{ which is equivalent to } \sigma_{\theta}^2 > \sigma_x^2. \text{ Also, (A10) is satisfied at } b = 1 \text{ if }
\]

\[
\left( \frac{\sigma_{\theta}^2}{\sigma_x^2 + \sigma_{\theta}^2} \right) \bar{\theta} < (1+g). \text{ Therefore if } \sigma_{\theta}^2 > \sigma_x^2 \text{ and } \bar{\theta} < \left(1 + \frac{\sigma_x^2}{\sigma_{\theta}^2}\right)(1+g)
\]

then, necessarily, \( a < 0 \).

Next, we establish sufficient conditions for \( a > 0 \). From (A10) it follows that \( a > 0 \) if and only if:

\[
(2-b) \left( \frac{b^2 \sigma_x^2}{\sigma_x^2 + b^2 \sigma_{\theta}^2} \right) \bar{\theta} > (1+g) \tag{A11}
\]

We have previously established that if \( \sigma_{\theta}^2 > \sigma_x^2 \) then the left hand side of (A11) is strictly decreasing at all equilibrium values of \( b \). Therefore if (A11) is satisfied at \( b = 1 + \alpha \) then this inequality must be satisfied at all equilibrium values of \( b \). At \( b = 1 + \alpha \),

The left hand side of (A11) = \( (1-\alpha) \left( \frac{(1+\alpha)^2}{\sigma_x^2 + (1+\alpha)^2} \right) \bar{\theta} \)

Then, if \( \frac{\sigma_x^2}{\sigma_{\theta}^2} < 1 \) a sufficient condition for \( a > 0 \) is:

\[
(1-\alpha) \left( \frac{(1+\alpha)^2}{1+(1+\alpha)^2} \right) \bar{\theta} > 1 + g
\]
Therefore if \( \sigma^2_\theta > \sigma^2_z \), and \( \overline{\theta} > \left( 1 + \frac{1}{(1 + \alpha)^2} \right) \left( \frac{1 + g}{1 - \alpha} \right) \) then, necessarily, \( a > 0 \).

**Proof of Proposition 5**

Since \( a = -m_0 + m_1 \overline{\theta} \), \( a + (b - 1)\overline{\theta} > 0 \) is equivalent to \( -m_0 + (m_1 + b - 1)\overline{\theta} > 0 \), or,

\[
\overline{\theta} > \frac{m_0}{m_1 + b - 1}.
\]

Substituting the expressions for \( \{b, m_0, m_1\} \) contained in (21), (23) and (24) gives, after considerable algebraic manipulation, (done in *Mathematica*),

\[
\frac{m_0}{m_1 + b - 1} = \frac{(1 + g) \beta \left[ 1 + \alpha (1 - 2 \beta (2 - \beta)) \right]}{(1 - \beta)(1 - \alpha \beta (2 - \beta))} \tag{A12}
\]

We verified with the aid of *Mathematica* that the right hand side of (A13) is strictly increasing in \( \beta \). Additionally, \( \sigma^2_\theta > \sigma^2_z \) and \( b > 1 \) imply that \( \beta_z \equiv \frac{\sigma^2_z}{\sigma^2_z + b^2 \sigma^2_\theta} < \frac{1}{2} \).

Therefore if at \( \beta = \frac{1}{2} \),

\[
\frac{(1 + g) \beta \left[ 1 + \alpha (1 - 2 \beta (2 - \beta)) \right]}{(1 - \beta)(1 - \alpha \beta (2 - \beta))} < \overline{\theta}
\]

then such an inequality holds at all values of \( \beta \) that are consistent with \( \sigma^2_\theta > \sigma^2_z \). Evaluating (A12) at \( \beta = \frac{1}{2} \) gives,

\[
\frac{m_0}{m_1 + b - 1} < \frac{2(1 + g)(2 - \alpha)}{4 - 3\alpha}.
\]
Thus, sufficient conditions for $a + (b-1)\bar{\theta} > 0$ are:

$$\sigma^2_\phi > \sigma^2_s$$ and $$\bar{\theta} > \frac{2(1+g)(2-\alpha)}{4-3\alpha}.$$
References


