Public Signals, Voluntary Disclosures, and Security Prices

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Abstract

This paper establishes that when a firm borrows, its manager interested in maximizing equity price is likely to disclose his private information more often if and only if the posterior default probability decreases following a public signal.

When both posterior mean and variance vary in the realized value of the public signal, we find that the posterior default probability is non-monotonic in public news. As a consequence, each of posterior probability of the manager’s discretionary disclosure and the firm’s equity price can be decreasing in the realized value of the positively correlated public news.

Applications of this trade-off between the posterior variance effect versus the mean effect generate several significant predictions about the impact of mandatory disclosures and analysts’ forecast consensus and dispersion on firms’ discretionary disclosures and equity prices. In particular, we predict that a mandatory disclosure requirement would actually decrease the voluntary disclosure likelihood for sufficiently large and small values of public news if prior default probability is less than 0.5.

Finally, most of above results apply qualitatively to an all-equity firm with limited liability for equityholders.
1 Introduction

Publicly listed firms often experience occurrence of a variety of significant public signals about them. Examples of such public signals with major economic significance would include firms’ own mandatory disclosures such as earnings reports, analysts’ forecasts, regulators’ actions and statements such as the Food and Drug Administration’s (dis) approval of a drug application and SEC’s investigations, news of a large proposed merger/acquisition and subsequent regulator’s approval or denial of such proposal (e.g., recent Broadcom’s bid for Qualcomm, etc.,) public news of data leaks caused by hacking (e.g., Equifax), social media responses to a firm’s proposed major plan, consumers’ responses to firms’ products and services such as the product market’s reception to a new model of Apple’s iphone, product recalls or hazards, development of a new product, process or technology (e..g., Boeing’s 787 or Tesla’s new model announcement), major law suits or judgements related to significant law suits, a highly visible or reputed manager’s departure from or joining a given firm, etc.

Acharya, DeMarzo, and Kremer’s, 2011, (ADK) theoretical analysis finds that any such public signals would not change firms’ voluntary or discretionary disclosure probabilities, regardless of whether such public signals are favorable or unfavorable and irrespective of the magnitude of their economic significance to the firms. ADK’s result arises from capital markets recalibrating their prior beliefs about distributions of firms’ cash flows based on such public signals in a manner that in equilibrium firms’ voluntary disclosure probabilities would not change.

However, there is a fair amount of empirical literature that does demonstrate firms’ voluntary disclosure strategies being influenced by a variety of public signals for a wide set of reasons. For instance, Lansford [2006] finds firms tend to make voluntary disclosures of favorable patent information more often following negative earnings surprises.1 This empirically documented phenomenon of voluntary disclosure behavior of firms being influenced by prior or concurrent public signals finds limited theoretical explanation.

ADK’s [2011] result of the indifference of firms’ voluntary disclosure proclivities to prior public news is obtained in a setting with unlimited liability for equityholders. A significant set of prior voluntary disclosure assumes similar unlimited liability for equityholders. However, it is a well-known legal fact that shareholders of incorporated firms do enjoy limited liability. So, do debtholders of all firms. This paper, therefore, examines whether such prior voluntary disclosure

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1 Other papers that find similar evidence of firms’ voluntary disclosure strategies being influenced by public news for several other reasons include firms reducing their disclosures in response to litigation events (Rogers and Van Buskirk, 2009), changes in market characteristics such as decreasing market uncertainty following earnings announcements mitigated by voluntary forecasts possibly leading to greater stock price volatility (Rogers, Skinner and Van Buskirk, 2009), and voluntary management forecasts being influenced by earnings reports (Rogers and Van Buskirk 2013.)
disclosure results can be extended to settings with limited liabilities.\(^2\) However, we know that the introduction of limited liability induces call-option-like features in equity price behavior.\(^3\) It, therefore, further appears natural that any examination of firms’ voluntary disclosure behavior in limited liability settings should also allow for markets to update both unknown mean and unknown variance to better understand firms’ optimal discretionary disclosure responses to such public signals and consequent securities price behavior.

We examine a setting in which a (possibly levered) firm’s cash flow is stochastically distributed with unknown mean and unknown variance. The markets update their posterior beliefs about both the mean and the variance based on a public signal which is correlated with the firm’s cash flow. The manager observes private information about the firm’s cash flows with some positive probability. One major objective is to understand how in the presence of limited liability, the manager’s discretionary disclosure decision responds to the public signal and how security prices vary in the public signal.

For a broad class of distributions, our analysis predicts that the public signal increases the levered firm’s discretionary disclosure probability if and only if the posterior default probability following the public signal decreases. This prediction is somewhat surprising for several reasons. First, the manager’s objective in deciding on whether to make a discretionary disclosure is only to maximize the firm’s expected equity price. Yet, his disclosure probability increases in the realized value of the public signal if and only if the posterior default probability decreases. As will be seen below, the posterior default probability is not monotone in the public signal. So, when the public signal is positively correlated with the firm’s cash flows, one might presume that greater public news should increase the manager’s likelihood of voluntary disclosure. Our result negates this intuition. Second, this result contrasts with ADK. [2011], who demonstrated that a public signal preceding a voluntary disclosure decision should not affect the voluntary disclosure probability because the original priors are recalibrated following the public signal in a manner that, with unlimited liability for equityholders, the manager’s voluntary disclosure probability would not change. In our setting, this inverse relation between the posterior default probability and the manager’s discretionary disclosure probability as a function of the realized value of the public signal

\(^2\)Sridhar and Magee [1996] examine earnings reports in the context of limited liability. Fischer and Verrecchia [1997] examine the impact of earnings reports on the behavior of equity and debt prices in the presence of such limited liability.

\(^3\)There is a large amount of empirical research which have examined non-linear price behavior and the impact of debt default probability on equity price. Papers that examine such price behavior include Freeman and Tse [1992], Subramanyan and Wild [1993], Das and Lev [1994], Dhalwal and Reynolds [1994], Hayn [1995], Bao and Datta [2014], Bischof et al. [2016], Campbell et al. [2014], Hope et al. [2016], etc.

However, most prior theoretical voluntary disclosure literature (including the one that examines such behavior in the presence of limited liability) focuses on updating only mean for a given known variance.
arises only because of the limited liability. Third, this result is independent of whether the posterior precision of the cash flow distribution is invariant with respect to the realized value of the public signal (e.g., as in the case of a normal distribution).

Another prediction that arises immediately from this inverse relation between the posterior default probability and the voluntary disclosure probability is that the probability of voluntary disclosure decreases in the amount of leverage. If the firm were not levered, this necessary and sufficient condition would still hold if the equity holders enjoy limited liability. The requirement that the equity price must be greater than or equal to zero in the presence of limited liability of equityholders would also yield the result that the firm’s voluntary disclosure probability varies in the realized value of the public signal.

We next proceed to examine a setting which allows for the market’s posterior beliefs about both the mean and the variance to change in the realized value of the public signal. We assume that the cash flow realized at the end of the game and the interim public signal are jointly distributed according to a multivariate student’s t distribution. Defining the public "news" as the magnitude of the surprise in the public signal, we find that with a normal distribution assumption for cash flows, the posterior equity value following the public signal is monotonically negatively related to the posterior default probability. We next proceed to examine a setting with a student’s t distribution that allows for the posterior variance also to change in the realized value of the public news. Our analysis predicts that the posterior equity price and default probabilities are positively correlated following sufficiently good and bad public news, provided the prior default probability is less than 0.5. That the equity price can actually increase in the posterior default probability may at first blush seem counterintuitive, but is best understood from examining the interaction between updating the mean versus variance based on the public signal. With fatter tails, a large surprise (either good or bad) in public news would increase the posterior variance in a significant manner. Consequently, when prior beliefs about the default probability are lower, then the posterior variance effect dominates the posterior mean effect for sufficiently good or bad public news, leading to the equity prices being positively correlated with posterior default probability.

However, where the prior default probability is greater than 0.5, even moderately bad public news can yield similar positive relation between the posterior default probability and equity prices. The intuition for this seemingly surprising result is that while the prior mean effect serves to increase the posterior default probability for moderately bad public news, the relatively low posterior variance is sufficient to increase the ‘option value’ effect embedded in equity prices on account of limited liability. In this way, prior beliefs about default probability continue to exert significant influence on the nature of relation between the posterior default probability and equity prices as a function of the the realized value of
These results concerning positive relation between the posterior default probability and the equity price vanish when the posterior variance does not change in the realized value of the public signal, as in the case of a normal distribution. This comparison highlights the important role played by the markets updating the variance in establishing the relation between (posterior) default probabilities and equity prices.

Our analysis further predicts that the posterior probability of a voluntary disclosure, following a public signal, always increases in the prior mean of the cash flow. However, whether the posterior probability of voluntary disclosure following the public signal would increase in the prior volatility of the cash flow depends on the magnitude of prior default probability. The posterior probability of voluntary disclosure actually decreases in the prior cash flow volatility if and only if the prior default probability is less than 0.5. We also predict that the posterior probability of voluntary disclosure decreases (increases) in the correlation between the public signal and the cash flow for relatively low (high) realized values of the public signal.

A major finding of our analysis is that the posterior default probability is non-monotonic in the public signal. When the prior default probability is less than 0.5, the posterior default probability is $U$-shaped; then, there exists a threshold level of public signal at which this posterior default probability reaches its minimum level, so that the posterior default probability is increasing and voluntary disclosure is decreasing for public signal values greater than this threshold. In contrast, when the prior default probability is more than 0.5, the posterior default probability is $hump$-shaped; then, there exists another threshold level of public signal at which this posterior default probability reaches its maximum level, so that the posterior default probability is increasing and voluntary disclosure is decreasing for public signal values less than this second threshold level. In this way, the prior default probability determines the shape of the posterior default probability following public signal, which shape in turn influences the posterior voluntary disclosure probability.

Moreover, our analysis predicts that either when the stochastic cash flows and public signals are jointly normally distributed or in the knife-edge case of the prior default probability being exactly equal to 0.5, the posterior default probability is monotonically decreasing and posterior voluntary disclosure probability is monotonically increasing in the realized value of the public signal.

We apply the conceptual findings from our general analysis to two major areas of interest: interpreting public signals as mandatory disclosures and as analysts’ forecasts. Our first set of results concerns with how the mandatory disclosures of a noisy signal of the firm’s cash flow affect the equity price behavior, particularly when not followed by (or a concurrent) voluntary...
disclosure. Our analysis predicts that the price for any given realized value of mandatory disclosure, followed by no voluntary disclosure, can increase in the prior mean of the cash flow and over regions of relatively less favorable values mandatory disclosures particularly when posterior default probability decreases. Such non-disclosure prices can also increase in the prior cash flow volatility when prior default probability is weakly greater than 0.5, but can decrease in the correlation between the cash flow and the public signal, particularly when the mandatory disclosure is relatively unfavorable. Further, our analysis predicts that the introduction of a mandatory disclosure requirement can actually decrease voluntary disclosure probabilities for sufficiently good and bad values of mandatory disclosures when prior default probability is less than 0.5. These results are driven by the markets updating both the unknown mean and variance based on the mandatory disclosures. These Einhorn [2005] also obtains similar predictions about how mandatory disclosures can reduce voluntary disclosures in the presence of multiple private correlated signals, one of which is required to be disclosed mandatorily.

Finally, when we view analysts’ forecast reports as a set of public signals, our analysis first predicts that the posterior mean of the firm’s cash flow conditioned on the entire set of analysts’ forecast reports can be reduced to a function of the analysts’ forecasts consensus. In other words, the analysts’ forecast consensus (as opposed to the entire vector of analysts’ reports) is sufficient to derive the posterior mean of the firm’s cash flows. Similarly, the analysts’ forecast consensus and dispersion together are sufficient (again, as opposed to the entire set of analysts’ reports) to determine the bayesian-updated posterior variance of the firm’s cash flows. Our analysis further predicts that the posterior default and posterior voluntary disclosure probabilities are non-monotonic in the analysts’ forecasts consensus; as a consequence, the posterior default probability actually increases and posterior voluntary probability actually increases in the consensus when prior default probability is greater than 0.5 and for sufficiently high values of consensus. Moreover, the firm’s posterior default probability is predicted to actually decrease (increase) in forecast dispersion if prior default probability is greater (less) than 0.5. Finally, we also predict that the firm’s posterior voluntary probability is predicted to actually increase (decrease) in forecast dispersion if prior default probability is greater (less) than 0.5.

While Sridhar and Magee [1996] examine manager’s opportunistic disclosure management in the context of a debt contract, they do not examine the role of a public signal in the presence of limited liability. Fischer and Verrecchia [1997] examine the impact of public signals in the presence of limited liability. In contrast, we examine how the presence of public signals affect the firm’s voluntary disclosure probabilities and generate a variety of predictions related to multiple applications of our setting. Prior literature that examines the
updating on variances include Subramanyam [1996], Jorgensen and Kirschenheiter [2003], Beyer [2009] and Heinle and Smith [2017]. Subramanyam [1996] was perhaps the first paper to show how posterior precision behaves for a fairly general class of distributions, and consequently, how stock returns exhibit non-linear properties in the surprise component of a public signal. While Jorgensen and Kirschenheiter [2003] examine the voluntary disclosure behavior of variances when such disclosure is costly, Beyer [2009] examines a setting in which the manager optimally manages both the forecast and earnings reports when the markets update both unknown mean and variance Heinle and Smith [2017] examine how a firm’s imperfect public signal about the variance of the cash flow would lead to variance uncertainty premium when the market knows the mean, but updates unknown variance. They proceed to demonstrate that the firm’s cost of capital decreases because the firm’s public signal reduces the variance uncertainty premium. In contrast to these papers, we examine how a manager’s voluntary disclosure strategy and price behavior are affected by a public signal when the market updates both unknown mean and variance.

2 The Model

Our goal is to study how public information affects the voluntary disclosures of firms in the presence of limited liability in general and, more specifically, for levered firms. Assume that at time $t = 0$, an incorporated firm must borrow $M$ to invest in a project which generates a random cash flow $X$ at time $t = 3$. Normalizing the expected rate of return for the debtholders to zero, the face value $\delta > 0$ is determined as the solution to the following participation constraint for lenders:

$$E [V_d (X)] = M,$$

where the value of the debt given $X$ is determined as

$$V_d (X) \equiv \max \{\min \{X, \delta\}, 0\}$$

Similarly, the equity value as a function of $X$ is given by

$$V_E (X) \equiv \max \{X - \delta, 0\}.$$

The expressions for equity and debt values above reflect the legal fact that owners of and

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4Throughout, we use capital letters to denote a random variable (e.g., $X$) and small letters to denote their realization (e.g., $x$).
lenders to incorporated firms enjoy limited liability. It follows that time equity and debt prices at the beginning of the game (i.e., at time \( t = 0 \)) are given by

\[
P_{0E}(\delta) = E[\max\{X - \delta, 0\}] \quad \text{and} \quad P_{0D}(\delta) = E[\max\{\min\{X, \delta\}, 0\}],
\]

respectively.

At time \( t = 1 \), a public signal \( Y \) about \( X \) is realized. At a later point in time, say, at time \( t = 2 \), we assume that (following Dye, 1985) with probability \( q \in (0, 1) \) the manager privately observes a perfect signal about the firm’s liquidating dividend, \( X \). If informed, the manager can publicly disclose his private information. We assume that any such voluntary disclosure must be truthful. Also, we assume that an uninformed manager cannot credibly convey the fact that he received no private information about \( X \). The goal of the manager is to choose disclosure of the realized \( X = x \) (denoted \( d, x \)) or non-disclosure (denoted \( nd \)) in order to maximize the market price of equity at time \( t = 2 \), \( P_{2E}(.) \). The public information set at time \( t = 2 \) is \( \mathcal{I} = \{y, x\} \), if the manager discloses, and \( \mathcal{I} = \{y, nd\} \), if the manager does not disclose.

Figure 1 below provides the time-line for our setting.

<table>
<thead>
<tr>
<th>( t = 0 )</th>
<th>( t = 1 )</th>
<th>( t = 2 )</th>
<th>( t = 3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>The firm borrows $M by entering into a debt contract with face value of debt ( \delta ).</td>
<td>A public signal ( Y ) is realized.</td>
<td>The firm’s manager observes a private signal ( X ) with prob. ( q \in (0, 1) ) and decides on a discretionary disclosure.</td>
<td>The firm realizes its liquidating dividend ( X ).</td>
</tr>
<tr>
<td>Equity price ( P_{1E}(Y) ) and debt price ( P_{1D}(Y) ) form.</td>
<td>Equity price ( P_{2E}(\mathcal{I}) ) and debt price ( P_{2D}(\mathcal{I}) ) form.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In a market populated by risk-neutral investors, the price of equity at time \( t = 2 \), \( P_{2E}(\mathcal{I}) \) given the public information set \( \mathcal{I} \) is

\[
P_{2E}(\mathcal{I}) = E[V_E(X) | \mathcal{I}],
\]

that is the posterior expected value of equity conditioned on all publicly available information at time \( t = 2 \).

We make the following assumption on the joint distribution of \((X, Y)\).
Assumption 1  The distribution of $X$ conditional on the realized $Y = y$ can be represented as follows:

$$X \mid (Y = y) = \mu (y) + \sigma (y) U,$$

where $\mu (\cdot)$ and $\sigma (\cdot)$ are deterministic functions and the random variable $U$ is zero-mean, continuously distributed with a positive density over the real line, and independent of $Y$.

That is, we assume that the realized public signal affects potentially both the posterior mean and variance of the liquidating value. For the moment, we do not impose any restrictions on $\mu (\cdot)$ and $\sigma (\cdot)$ other than $\sigma (y) > 0$ for all $y$. This condition guarantees that observing $Y$ never resolves all the uncertainty about $X$. The representation of the posterior distribution in (1) is rather general, and can be recovered from various (commonly used) joint distributions of $(X, Y)$. Two important examples are the multivariate normal and multivariate Student’s $t$ distribution. Our first major result, Proposition 1, relies only on the broad distributional structure in assumption (1).

3 Default probability and voluntary disclosures

The objective of this section is to understand how the limited liability structure affects the manager’s equilibrium discretionary disclosure behavior. Fix a public signal $Y = y$. It is straightforward to establish that there exists a disclosure threshold $\hat{x} (y)$ such that the set of realizations of $X$ that are voluntarily disclosed in equilibrium must be an interval of the form $[\hat{x} (y), \infty)$. This (usual) result follows from the disclosure payoff, $P_{2E} (y, x)$, weakly increasing in the realization $X = x$ while the non-disclosure payoff, $P_{2E} (\{y, nd\})$, is independent of $x$. Further, note that the disclosure threshold $\hat{x}$ must be such that $\hat{x} > \delta$. Otherwise, the disclosure payoff for $x < \delta$ would be zero even though the non-disclosure payoff is strictly positive. This latter result follows from a strictly positive posterior weight to the possibility of the manager being uninformed and that posterior expected residual claims of equityholders conditioned on the manager not being informed are strictly positive, i.e., $\Pr(\text{the manager not being informed} \mid nd) \times E \left[ (X - \delta)_+ \mid Y = y, \text{the manager not informed} \right] > 0$.

Therefore, the disclosure threshold $\hat{x}$ must satisfy the indifference condition:

$$\hat{x} (y) - \delta = \frac{(1 - q) E \left[ (X - \delta)_+ \mid Y = y \right] + q \Pr \left[ X < \hat{x} (y) \mid Y = y \right] E \left[ (X - \delta)_+ \mid Y = y, x < \hat{x} (y) \right]}{(1 - q) + q \Pr \left[ X < \hat{x} (y) \mid Y = y \right]},$$

where $f_{X \mid Y = y} (x)$ denotes the density function of $X \mid Y = y$ which is used to determine the above posterior expectations. The left hand side of the above equation represents the payoff from disclosing $X = \hat{x} (y)$ and the right hand side denotes the date-2 price given the public
signal \( Y = y \) and the firm’s non-disclosure, \( P_{2E}(\{y, nd\}) \).

For any given face value of debt \( \delta \), define the posterior default probability following the public signal \( Y = y \) as \( \Pr (X < \delta|y, \delta) \).

When equity holders have unlimited liability, Acharya, Demarzo and Kremer [2011] show that any public signal merely recalibrates the prior beliefs, and hence, the manager’s disclosure likelihood does not vary in the realized value of the public signal \( Y \). In contrast, Proposition 1 below predicts that the manager’s voluntary disclosure likelihood is inversely related to the posterior default probability, where such posterior default probability is conditioned on the public signal \( Y \). In other words, even though the manager’s objective is to maximize the firm’s equity price at time \( t = 2 \), a necessary and sufficient condition for the manager’s disclosure probability to increase is a decreased posterior default probability. All proofs are in the Appendix.

**Proposition 1** The realized public signal \( Y = y \) increases the posterior probability of voluntary disclosure of the manager’s private information \( X \) if and only if it decreases the posterior default probability, \( \Pr [X < \delta|Y = y] \).

Here it must be noted that the firm need not be a levered firm to obtain the above result. In other words, Proposition 1 result would hold even for an all-equity firm, where the legal construct of limited liability for equityholders of incorporated firms would mean \( \delta \equiv 0 \). For any given public signal \( Y = y \), define the standardized disclosure threshold and face value by

\[
\tilde{x}(y) = \frac{\tilde{x} - \mu(y)}{\sigma(y)}, \quad \tilde{\delta}(y) = \frac{\delta - \mu(y)}{\sigma(y)}
\]

The intuition for Proposition 1 result follows from noting that the derivatives of \( \tilde{x}(y) \) and \( \tilde{\delta}(y) \) in the realized value of the public signal \( y \) have the same sign regardless of how the functions \( \mu(y) \) and \( \sigma(y) \) vary in \( y \) subject to our assumption that \( \sigma(y) > 0 \) for all \( y \). Using the distributional structure in (1) in the indifference condition (2), the result in Proposition 1 follows.

One might have imagined that given the manager’s objective function of maximizing the equity price \( P_{2E}(\cdot) \), the posterior disclosure probability would also be directly influenced by posterior expected residual claims of equity holders after paying the debtholders’ claims for any given public signal \( Y = y \). However, Proposition 1 establishes the somewhat surprising result that the manager’s discretionary disclosure behavior may seem to mimic more the debtholders’ interests given the result that increasing disclosure likelihood as the posterior default probability decreases.
Acharya et al., (2011) point out that the release of a public signal prior to the manager’s discretionary disclosure decision has a twofold effect on the probability of information withholding: a distribution effect and a threshold effect. The former effect consists in the fact that, for a fixed threshold \( \hat{x} \), the realized \( y \) changes the posterior mean \( \mu(y) \) and volatility \( \sigma(y) \) of \( X \), thereby changing the probability of the firm withholding. The latter is an effect that occurs because a different posterior distribution of \( X \) implies a different equilibrium threshold; in other words, \( \tilde{x}(y) \) itself is a function of the public news. In voluntary disclosure models with unlimited liability such as Acharya et al. (2011), the two effects exactly offset each other. Here, on the contrary, the distribution effect dominates the threshold effect. The disclosure decision is taken to maximize the equity value, \( \max \{ X - \delta, 0 \} \). From a statistical point of view, the residual claim to equity holders is a censored version of the random variable \( X \). Therefore, equity prices respond less starkly to the public news than does the distribution of \( X \), leading to the result that a greater posterior default probability discourages voluntary disclosures.

Acharya and Johnson [2007] find a greater incidence of insider trading in those credit default swap (CDS) markets for those securities which are characterized by greater credit risks. This empirical finding is consistent with our prediction of equilibrium disclosure behavior. A higher posterior default probability would more likely lead to a greater credit risk. Less frequent voluntary disclosures under such circumstances would facilitate more aggressive insider trading by the manager to take advantage of the consequent greater information asymmetry in the market. Further, Proposition 1 result provides one possible theoretical explanation for the empirical finding in Drucker and Puri [2009] of a negative association between the likelihood of a loan sale by banks and the distance-to-default of the client firm. The bank that had lent to this firm initially at \( t = 1 \) is more likely to obtain superior private information from monitoring the client following a higher posterior default rate. Consequently, the bank is more likely to sell its asset in the secondary market more often given the greater likelihood of non-disclosure by the firm. In this way, the firm’s increasing reluctance to make voluntary disclosures in the posterior default probability would encourage the bank to sell its asset more frequently in the secondary market.

A direct consequence of the result in Proposition 1 generates the following prediction.

**Remark 1** The posterior probability of voluntary disclosure is decreasing in the amount of borrowing \( M \) for every realized value of the public news, \( Z = z \).

As the amount of borrowing \( M \) increases, ceteris paribus, the face value of \( \delta \) would increase. This in turn causes the posterior default probability to increase, thus leading to lower frequency of voluntary disclosures.
An immediate implication of the above corollary is that following any given public signal, an all-equity firm is more likely to make a voluntary disclosure than any levered firm.

4 Impact of public signal on default probability, security prices and voluntary disclosures

Recollect that the result in Proposition 1 only relied on the distributional structure in assumption (1). To model the notion of an economically significant public signal that would allow the manager and the markets to update both the mean and the variance of the firm’s cash flows, we make the following distributional assumption.\(^5\)

**Assumption 2** \((X, Y)\) are jointly distributed according to a bivariate \(t\) distribution (denoted \(t_2\)) with \(k \geq 3\) degrees of freedom,\(^6\)

\[
\begin{pmatrix} X \\ Y \end{pmatrix} \sim t_2 \left( \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \begin{pmatrix} \Sigma_X & \Sigma_{XY} \\ \Sigma_{XY} & \Sigma_Y \end{pmatrix}, k \right),
\]

where \(\rho_{XY} = \frac{\Sigma_{XY}}{\sqrt{\Sigma_X \Sigma_Y}}\) is such that \(0 < \rho_{XY} < 1\).

As is well known (DeGroot, 1970), the variance-covariance matrix of the joint \(t\) distribution (of the firm’s cash flows \(X\) and the public signal \(Y\)) is given by

\[
\frac{k}{k-2} \begin{pmatrix} \Sigma_X & \Sigma_{XY} \\ \Sigma_{XY} & \Sigma_Y \end{pmatrix}.
\]

Therefore, the inequality \(\rho_{XY} > 0\) requires that the liquidation value and the public signal are positively correlated. This assumption allows us to interpret a higher realization of the public signal as better news.\(^7\) The correlation between \(X\) and \(Y\), denoted \(\rho_{XY}\), is assumed to be less than one in absolute value, so that \(Y\) is not a perfect signal of \(X\).

Define as “news” \(Z = \Sigma_Y^{-\frac{1}{2}}(Y - \mu_Y)\) as the magnitude of the surprise in the public signal \((Y - \mu_Y)\) normalized by its standard deviation, \(\sqrt{\Sigma_Y}\).

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5If the equityholders were to be subject to unlimited liability, then even with our student’s \(t\) distribution assumption we would get the linearity of equity price in public signal \(Y\) and the probability of voluntary disclosure being independent of the public signal, results that are well known in other contexts, e.g., Acharya, DeMarzo and Kremer [2011].

6We require at least three degrees of freedom to ensure that the variance-covariance matrix is well-defined. Further, note that in the limiting case, as the degrees of freedom \(k \to \infty\), the \(t\) distribution approaches a normal distribution.

7Assuming \(\rho_{XY} > 0\) is also without loss of generality, because if the correlation were negative we would adopt the opposite convention that a lower realization of \(Y\) is better news.
Remark 2 The random variable $X|Y = y$ can be represented as in (1):

$$X|Z = z = \mu(z) + \sigma(z) U_{k+1}, \quad (3)$$

where $U_{k+1}$ denotes a standard univariate $t$ distribution with $k + 1$ degrees of freedom,

$$\mu(z) = \mu_X + \rho_{XY} \sigma_X z, \quad \text{and} \quad \sigma(z) = \sigma_X \left(1 - \rho_{XY}^2\right) \frac{k + z^2}{k + 1}.$$

In the above, $\sigma_X \equiv \sqrt{\sum_X}$ parametrizes the standard deviation of the liquidation value (for fixed degrees of freedom $k$).

Using these definitions, the posterior expected equity value conditioned on the normalized values of the public signal are

$$E[V_E(X)|Z = z] = \sigma(z) \int_{\tilde{\delta}}^{\infty} \left(u - \tilde{\mu}\right) g_k(u) du, \quad (4)$$

where $\tilde{\delta} \equiv \frac{\delta - \mu(z)}{\sigma(z)}$ and $\tilde{\mu} \equiv \frac{\mu(z)}{\sigma(z)}$ are “standardized” versions of the face value of the debt and of the mean, respectively, and $g_k(\cdot)$ is the density of the standard univariate $t$ distribution for given $k$ degrees of freedom. Expressions (4) and (??) are derived in the proof of Proposition 2. Also, the probability of default conditional on public news after normalization can be expressed as:

$$\Pr[X < \delta|Z = z] = G_k(\tilde{\delta}).$$

We next generate predictions of how date-1 equity prices relate to posterior default probabilities as functions of both prior beliefs as at $t = 0$ and date-1 public signal $Y$.

**Proposition 2** (a) When only the posterior mean (and not the posterior variance) of cash flow $X$ changes in the realized value of the public signal $Y$ (i.e., when $k \to \infty$):

The posterior equity value at $t = 1$, $P_{1E}(z)$, is negatively related to the posterior default probability at $t = 1$.

(b) When both the posterior mean and posterior variance of cash flow $X$ change in the realized value of the public signal $Y$ (i.e., when $k < \infty$):

(b.i) If the prior default probability is lower than 0.5, then the posterior equity price at $t = 1$, $P_{1E}(z)$, and the posterior default probability are positively related for

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The actual standard deviation is $\sqrt{\frac{k}{k-2}}\sigma_X$, because the marginal of $X$ is a $t$ distribution, so $\sigma_X$ is the standard deviation of $X$ only in the limiting case of the normal distribution ($k \to \infty$).
sufficiently good and bad public news \( z \); and

(b.ii) If the prior default probability is greater than 0.5, the posterior equity value at \( t = 1 \), \( P_{1E}(z) \), and the posterior default probability are positively related for moderately bad public news \( z \).

The intuition for the above predictions follows. Recollect that when the degrees of freedom \( k \to \infty \), the student’s t-distribution approaches the normal distribution, in which case the posterior variance conditioned on the realized value of public news \( Z \) is independent of such \( Z \). So, the posterior distribution shifts to the right in the sense of first order stochastic dominance leading to increases in posterior expected equity value and a decrease in the posterior default probability. Hence, we get the result in Proposition (2a).

However, when \( k < \infty \), the posterior mean and variance move in fairly complex ways in public news. As a consequence, the relationship between posterior equity prices and posterior default probability are not as simple as in the case of a normal distribution updating. First, observe from Remark (2) that the posterior variance increases in the magnitude of publicly disclosed surprise, \( z \). When prior beliefs are optimistic (i.e., when prior default probability is lower than 0.5), the posterior mean effect is dominated by posterior variance effect both for sufficiently large favorable and unfavorable public news. Hence, the posterior volatility effect can dominate the posterior mean effect for sufficiently high values of public news \( z \), with the result that both the ‘option-type’ characteristic of equity value and the posterior probability of default can increase at the same time in \( z \). This dominating posterior volatility effect yields the results in Proposition (2b.i).

The result in Proposition (2b.ii) also follows from how the posterior mean versus variance effect trade-off is influenced by prior beliefs about default probability. With a high prior default probability, the positive relation between posterior equity price, \( . P_{1E}(z) \), and posterior default probability occurs only for moderate values of public news, a region over which the posterior variance effect is muted.

A major implication of results in part (b) of Proposition (2) is that even though markets update their posterior beliefs based on the public signal \( Y \) at \( t = 1 \), the relation between security prices and posterior default probability continue to be significantly influenced by prior beliefs (at \( t = 0 \)) about default likelihood. This again is a distinct feature of our setting because the markets use the public signal \( Y \) to update their priors about both the mean and the variance. In contrast, note that in the setting of part (a) of Proposition (2) when the markets use the public signal to update only their prior beliefs about the mean, the prior beliefs cease to play any role in determining the security prices’ relation to posterior default probability once the markets make their Bayesian updating.
Figure 1 below illustrates that, as one would expect, when the cash flow $X$ and the public signal $Y$ follow a joint normal distribution (as in Fischer and Verrecchia, 1997), the posterior default probability monotonically decreases in the positively correlated public signal $Z$ and the posterior equity price following the public signal decreases monotonically in such signal regardless of the prior default probability value. A similar behavior occurs with joint $t$ distribution only in the knife-edge case of prior default probability is exactly equal to 0.5. This figure assumes a specific form of the public signal $Y = X + \epsilon$.

![Graph showing monotonic behavior of posterior default probability and equity price, $P_{1E}(z)$](image)

Assume $Y = X + \epsilon$ and $q = 0.8, \mu_x = 0, \delta = 1.5, \sigma_x = 0.5, \text{and} \sigma_e = 0.5$.

The following figures 2(a) and 2(b) revert to the more general form of a noisy, but positively correlated public signal as represented in Assumption 2 above. Figure 2(a) illustrates Proposition (2b.i) result which deals with the setting of prior default probability being less than 0.5 (i.e., when $\delta < \mu_x$). In this case, figure 2(a) illustrates the equity price $P_{1E}(z)$ decreases along with posterior default probability for relative bad public news $z$ and also increases along with posterior default probability in the public signal $z$ beyond the threshold $z^*$, the point at which the posterior default probability attains its minimum. The posterior variance increasing significantly in sufficiently positive public news $z$ contributes to the increase the posterior default, while at the same time the call option-like feature of the equity price increases in the volatility particularly given that the low prior default probability mitigates the rate of increase in the posterior default probability in $z$. Similar intuition can be adapted to the case of sufficiently low values of $z$ as well because the posterior variance...
significantly increases for sufficiently low values of $z$ as well.

Figure 2(a): Prop. 2(bi) result when prior default probability is less than 0.5

Figure 2(b): Prop. 2(bii) result - when prior default probability is greater than 0.5

Figure 2 (a) : $k = 4, q = 0.8, \mu_x = 2, \delta = 1.5, \sigma_x = 0.5$ and $\sigma_\varepsilon = 1.8$

Figure 2 (b) : $k = 4, q = 0.8, \mu_x = 0, \delta = 0.1, \sigma_x = 0.5$ and $\sigma_\varepsilon = 0.5$

In contrast, Figure 2(b) illustrates Proposition (2b.ii) result when the prior default probability is greater than 0.5 (i.e., when $\delta > \mu_x$). Here, one can see that both the posterior default probability and the posterior equity price $P_{1E}(z)$ decrease in moderately negative public news $z$. Given the high prior default probability, even a moderately favorable public news $z$ is sufficient to reduce the posterior default probability, whereas the posterior variance for such moderate public news $z$ is not sufficient to increase the value of call option feature embedded in equity prices to compensate for the decreasing ‘mean effect’ implied by the bad news $z$. In this way, figures 2(a) and 2(b) together illustrate the seemingly puzzling result that equity prices can co-move in the same direction along with the posterior default probability over different regions of the realized public news $z$, depending on the prior beliefs about the default probability.

The next Proposition generates predictions about the firm’s discretionary disclosure proclivities as a function of some significant parameters of our setting.

**Proposition 3** The posterior probability of voluntary disclosure is:
(a) increasing in the prior mean of the cash flow $\mu_x$;
(b) decreasing (increasing) in the prior volatility of the cash flow, $\sigma_X$, if the prior default
probability is less (greater) than 0.5; and
(c) increasing (decreasing) in the correlation between the public signal and the liquidation
value, \( \rho_{XY} \), for relatively high (low) realized public news \( z \).

The intuition for these comparative statics results follows. The result in Proposition (3.1) is obvious. As the prior mean \( \mu_x \) of the cash flow increases, the posterior mean value also increases and greater voluntary disclosure incentives to confirm such posterior beliefs arise naturally.

The intuition behind Proposition (3.2) is more complex. Observe that the default probability is

\[
\frac{\delta - \mu_X - \rho_{XY}\sigma_X z}{\sigma(z)} \propto \frac{\delta - \mu_X - \rho_{XY}\sigma_X z}{\sigma_X} = \frac{\delta - \mu_X}{\sigma_X} - \rho_{XY}z.
\]

The above expression implies that the prior cash flow variance has an effect on voluntary disclosure only through the change in the prior default probability on account of the public news.

First, it can be established that in the knife-edge case of prior default probability being exactly equal to 0.5, the posterior probability of voluntary disclosure is independent of the cash flow volatility \( \sigma_x \). In this case, the prior distribution is symmetric around zero; so, though increasing variance thickens the tails, the distribution symmetry remains unchanged yielding the result that voluntary disclosure probability being independent in cash flow volatility \( \sigma_x \). When \( \delta - \mu_x > 0 \), i.e., when prior default probability is greater than 0.5, then increasing the cash flow volatility is good news because of a higher likelihood of cash flows being greater than the face value of debt \( \delta \). So, a lower posterior default probability increases the likelihood of voluntary disclosures.

Finally, the intuition behind Proposition (3.3) is straightforward. As the correlation between the cash flow \( X \) and the public signal, \( Y; \rho_{XY} \) increases, a good public news provides greater incentives for the firm to confirm the public signal with his own credible voluntary disclosures and vice versa.

While Proposition (3.) establishes monotonic voluntary disclosure behavior in prior parameters such as mean, variance and correlation coefficient, the next Proposition predicts that the public signal at \( t = 1 \) can induce non-monotonic voluntary disclosure behavior by the firm as a function of prior default probability.

**Proposition 4** (a) If the prior default probability is less than 0.5, then there exists a threshold \( z^* \) for the public news \( Z \) such that:

(ai) the posterior default probability is decreasing and probability of voluntary disclosure is increasing in the public news \( z < z^* \); and
(aii) the posterior default probability is increasing and probability of voluntary disclosure is decreasing in the public news $z > z^*$. 

(b) If the prior default probability is greater than 0.5, then there exists a threshold $z^{**}$ for the public news $Z$ such that:

(bi) the posterior default probability is increasing and posterior probability of voluntary disclosure is decreasing in the public news $z < z^{**}$; and

(bii) the posterior default probability is decreasing and posterior probability of voluntary disclosure is increasing in the public news $z > z^{**}$.

Proposition (4) highlights the result that the effect of public news on firms’ subsequent voluntary disclosure behavior depends significantly on prior beliefs about default probability. When the prior default probability is less than 0.5 (i.e., when $\delta < \mu_z$), Proposition (4) shows that the posterior default probability is actually increasing in the public news $Z$ for sufficiently good news, i.e., for $z > z^*$. When the prior default probability is low, a sufficiently favorable surprise in public signal improves the posterior perception about the cash flow $X$ significantly, thus diminishing the incentives of the firm to make a voluntary disclosure for moderate values. Similarly, with a low prior default probability belief, the firm has incentives to make voluntary disclosures for low public news to improve the firm’s market price following voluntary disclosures at $t = 2$.

These voluntary disclosure incentives flip when the prior default probability is high. The firm needs to confirm good public news with its own voluntary disclosures, thus increasing the likelihood of voluntary disclosures for sufficiently favorable public news. In this way, the prior default probability actually serves as a de facto switch that turns the voluntary disclosure as an informational ‘compliment’ to the public news when prior default probability is high and a ‘substitute’ for the public signal with low prior default probability. This significant role of prior beliefs is driven by the market updating both the mean and variance based on public news.

Figure 3 below illustrates that the cash flow $X$ and public signal $Y$ have joint normal distribution the posterior disclosure probability is monotonically increasing, while posterior default probability is monotonically decreasing in the public news $z$ regardless of the prior default probability value. A similar behavior occurs with joint $t$ distribution only in the knife-edge case of prior default probability is exactly equal to 0.5.
Figure 3: Monotonic behavior of posterior default and voluntary disclosure probabilities

Assume $Y = X + \epsilon$, $q = 0.8$, $\mu_x = 2$, $\delta = 1.5$, $\sigma_x = 0.5$, and $\sigma_\epsilon = 1.8$

Figure 4 (a) below illustrates the setting in Proposition 4 (a) in which prior default probability is less than 0.5 (i.e., when $\delta < \mu_x$). Here, $z^*$ refers to the value of $z$ at which the posterior default probability attains its minimum. For values $z < z^*$, the posterior default probability decreasing and the posterior voluntary disclosure probability increasing in public news $z$ is intuitive as in figure 3 above. However, for the public news $z > z^*$, the posterior disclosure probability is actually decreasing in $z$ because of the greater posterior volatility effect increases the posterior default probability in a significant enough manner to overcome any expected benefits from disclosure.
Corollary 1 The posterior probability of voluntary disclosure at $t = 2$ is monotonically increasing and the posterior default probability is monotonically decreasing in the realized value of the public news $z$ when:

(a) the prior default probability is exactly equal to 0.5, given that the posterior variance varies in the realized value of public signal $Y$ (i.e., when the degrees of freedom $k < \infty$); or
(b) regardless of the prior default probability value, if the posterior variance does not vary in the realized value of public signal \( Y \) (i.e., when the degrees of freedom \( k \to \infty \)).

Part (a) of Corollary (1) deals with the knife-edge case of the prior default probability being exactly equal to 0.5. In this case, the markets updating based on both the mean and the variance would still lead to monotonically increasing disclosure likelihood because any ‘residual memory’ from prior beliefs post updating does not play any role in determining the disclosure probability. Part (b) of Corollary (1) obtains a similar monotonically increasing disclosure probability result because when \( k \to \infty \), we obtain the normal distribution which does not allow for posterior variance to vary in the realized value of the public news \( Y \). Therefore, the information content of public news provides only a ‘compliment’ role for the firm’s voluntary disclosures, with no scope for the ‘substitute’ role, implying that the firm’s incentives to disclose increase everywhere in the public news \( Y \).

The next Corollary seeks to provide one possible explanation for the gap between the empirical literature (such as Miller, 2005) findings about better public news inducing greater voluntary disclosures and prior theoretical work (such as ADK, and Einhorn, 2005)

**Corollary 2** The probability of voluntary disclosure is higher conditional on relatively good public news \( Y \) than conditional on relatively bad public news.

In prior disclosure models with unlimited liability (ADK and Einhorn 2005), public information does not affect subsequent voluntary disclosure probability. But, Miller [2005] empirically finds that good public news actually prompts greater voluntary disclosures. A combination of limited liability and posterior variance varying in the public news yields the result in Corollary (2).

## 5 Applications

All the results above predict how, in the presence of limited liability, the voluntary disclosure strategy of a firm is affected by the realized public news and its statistical properties. The above results do not hinge on a particular source of public information, which can be an analyst report, a competitor’s announcement, or a mandatory disclosure of the firm itself. We next focus on two specific cases – the interaction between mandatory and voluntary disclosures and analyst forecasts – and expand the set of empirical predictions generated by our baseline model.
5.1 Mandatory and voluntary disclosures

Consider earnings announcements, when companies disclose their current earnings and, sometimes, contextually make a forecast about future earnings. By interpreting $Y$ as a mandatory disclosure, our baseline model can be immediately employed to study how the propensity to make a voluntary disclosure depends on the characteristics of the mandatory environment. Therefore, all prior findings can be rephrased in terms of mandatory disclosure of $Y$ and (either concurrent or following) voluntary disclosure of $X$.

Prior results (Proposition 2) primarily focused on properties of $P_{1E}(z)$, the interim equity price at $t = 1$ following the surprise element in mandatory disclosure (hereafter, simply referred to as mandatory disclosure) $z$, but before the firm has an opportunity to make a voluntary disclosure. In contrast, the following Proposition predicts the behavior of $P_{2E}(\{z, nd\})$, the equity price at time $t = 2$ following the mandatory disclosure of $z$ and no voluntary disclosure ($nd$) by the firm.

**Proposition 5** Let $z^*$ and $z^{**}$ denote the cutoffs for the mandatory disclosures at $t = 1$, as identified by Propositions 4(a) and 4(b), respectively. The stock price following the realized mandatory disclosure $z$ and no voluntary disclosure, $P_{2E}(\{z, nd\})$, is:

(a) increasing in the prior mean of cash flow, $\mu_X$;

(b) decreasing in the correlation between the cash flow and the mandatory disclosure $\rho_{XY}$, if the realized mandatory news $z$ is relatively low;

(c) increasing in the mandatory disclosure $z \in (0, z^*)$, if the prior default probability is less than 0.5;

(d) decreasing in the mandatory disclosure $z < z^{**}$ and increasing in $z > 0$, if the prior default probability is greater than 0.5;

(e) increasing in the prior volatility of the cash flow $\sigma_x$ if the prior default probability is greater than or equal to 0.5; and

(f) increasing in mandatory disclosure $z > 0$ if the prior default probability is equal to 0.5.

The result in Proposition 5(a) is intuitive. As the prior mean $\mu_x$ of the cash flows increases, Proposition 5(a) states that for any given mandatory disclosure $z$, the market price following no voluntary disclosure by the firm, $P_{2E}(\{z, nd\})$, would also increase. Recollect that the probability of the manager receiving private information is independent of the realized value of cash flow $X$, and hence, the market would reward the firm with a better non-disclosure price $P_{2E}(\{z, nd\})$ as the prior mean $\mu_x$ increases. As the correlation between the cash flow and mandatory disclosure, $\rho_{xy}$, increases, a low mandatory disclosure $z$ would only further confirm the market’s posterior beliefs about the cash flow $x$ also being low;
hence, any non-disclosure by the firm is viewed by the market more skeptically, leading to a lower price $P_{2E}(\{z, nd\})$. This establishes the result in Proposition 5(b).

Recollect from prior discussion that when the prior default probability is less than 0.5 (i.e., when $\delta < \mu_x$), $z^*$ represents the value at which the posterior default probability reaches its minimum. Proposition 5(c) states that the equity price following no voluntary disclosure, $P_{2E}(\{z, nd\})$, is increasing in $z \in (0, z^*)$ because of a combination of two factors: (a) conditioned on the manager not being privately informed, the value of equity claims are increasing in $z \in (0, z^*)$; and (b) the posterior default probability is also decreasing in $z \in (0, z^*)$ given that the prior default probability is less than 0.5.

Next, recollect that $z^{**}$ represents the point at which the posterior default probability attains its maximum value. When the prior probability is greater than 0.5 (i.e., when $\delta > \mu_x$), Proposition 5(d) predicts that the equity price following no disclosure, $P_{2E}(\{z, nd\})$, to decrease in $z < z^{**}$ partly because of increasing default probability over that region.

Proposition 5(e) predicts that a higher cash flow volatility $\sigma_x$ would increase the non-disclosure equity price $P_{2E}(\{z, nd\})$ for any given $z$ provided the prior default probability is greater than or equal to 0.5. This result partly follows from the fact that conditioned on the manager not being privately informed, the value of ‘option like’ feature of equity of a levered firm increases in the cash flow volatility $\sigma_x$.

Finally, Proposition 5(f) highlights the result that when the prior default probability is exactly equal to 0.5, the monotonically increasing no-disclosure prices $P_{2E}(\{z, nd\})$ reflect the setting with a normal distribution (i.e., when $k \to \infty$) when the posterior variance does not vary in the mandatory disclosure $z$.

The above Proposition generated a variety of predictions about how a given level of equity price following non-disclosure, $P_{2E}(\{z, nd\})$, would change in a variety of parameters. We next proceed to generate additional predictions about how the stock price reaction following non-disclosure. Define the scaled price reaction between date 1 and date 2 as

$$\Delta (z, nd) \equiv \frac{P_{2E}(\{z, nd\}) - P_{1E}(z)}{\sigma (z)}. \tag{5}$$

From our indifference condition for disclosure threshold, it is clear that this price reaction to the firm’s non-disclosure is always negative for any given mandatory disclosure $z$ because non-disclosure induces the market’s skeptical beliefs. The next generation highlights non-monotonic behavior of this stock price reaction following non-disclosure by the firm following the firm’s mandatory disclosure $z$.

**Proposition 6** The negative equity price reaction to the firm’s non-disclosure relative to the price following prior mandatory disclosure $z$ appropriately scaled, $\Delta (z, nd)$, will be:
(a) U-shaped in the realized mandatory disclosure, $z$, if the prior default probability is less than 0.5;
(b) hump-shaped in $z$, if the prior default probability is greater than 0.5;
(c) more negative if the prior mean, $\mu_X$, increases;
(d) less negative (more negative) in the prior volatility, $\sigma_X$, if the prior default probability is less (greater) than 0.5; and
(e) increasing (decreasing) in the correlation between the public signal and the liquidation value, $\rho_{XY}$, for relatively high (low) prior mandatory disclosure value $z$.

The intuition for the results in Proposition 6 follow closely to that of Proposition 4. It can be shown that the stock price reaction to non-disclosure, $\Delta(z, nd)$, varies in mandatory disclosure $z$ at a rate that is proportional to the sensitivity of the normalized face value, $\tilde{\delta} = \frac{\delta - \mu(z)}{\sigma(z)}$ in $z$. Therefore, combining the feature that the stock price reaction to non-disclosure, $\Delta(z, nd)$, is negative for all values of mandatory disclosure $z$, together with the reasoning behind Proposition 4 results yields the above comparative statics predictions in Proposition 6.

The above results are further clarified by the following set of illustrations. Figures 5 (a) and 5 (b) depict that regardless of prior default probability value: (a) the stock price reaction is non-monotonic in the quality of mandatory disclosures; and (b) further, as one would expect, higher quality of mandatory disclosures cause the stock price reaction $\Delta(z, nd)$ to non-disclosure more negative for less favorable mandatory disclosures $z$ and more muted (i.e., less negative) for favorable mandatory disclosures $z$ than lower quality of mandatory disclosures.
Figures 5 (c) and 5 (d) highlight the results from Proposition 6 (d) that a higher cash flow volatility has a muted (more negative) effect on the negative stock price reaction to non-disclosure, $\Delta(z, nd)$, when the prior default probability is less (greater) than 0.5. With a higher (lower) prior default probability, an increase in cash flow volatility causes the market to attribute the firm’s non-disclosure to a greater (lower) chance of the firm receiving less favorable private information, which yield the relation illustrated in figures 5 (c) and 5 (d). Further, these figures illustrate that for any given cash flow volatility, the negative stock
price reaction to non-disclosure, $\Delta (z, nd)$, is non-monotone.

Figure 5(c) : Effect of cash flow volatility on stock price reaction $\Delta (z, nd)$

Figure 5(d) : Effect of cash flow volatility on stock price reaction $\Delta (z, nd)$

Figure 5(e) : $q = 0.8, k = 4, \mu_x = 2, \delta = 1.5, \text{ and } \sigma_x = 1.8$

Figure 5(f) : $q = 0.8, k = 4, \mu_x = 0, \delta = 1.5, \text{ and } \sigma_x = 1.8$

Figure 5(e) illustrates that the negative stock price reaction to non-disclosure, $\Delta (z, nd)$, becomes stronger (i.e., more negative) when prior default probability is less than 0.5 than when more than 0.5. Figure 5(f) depicts a similar effect with a higher prior mean than with a lower one. In both cases, when the market has a more favorable prior belief (either in terms of a lower default probability or a higher mean), the firm’s non-disclosure imposes a
greater level of market skepticism, producing a starker negative response to non-disclosure.

Figure 5(e) : Effect of prior default probability on stock price reaction \( \Delta (z, nd) \)

Figure 6 (a) and 6 (b) illustrate the role of mandatory disclosure requirement in influencing subsequent voluntary disclosure probability. A comparison of figures 6 (a) and 6 (b) reveals three interesting features. First, when prior default probability is less than 0.5, figure 6 (a) below illustrates that posterior voluntary disclosure probability in the presence of mandatory disclosures is mostly lower than such discretionary disclosure probability in the absence of such mandatory requirement, except for a small neighborhood around the maximum point of posterior voluntary disclosure probability in the presence of the mandatory disclosure requirement.

The previous discussions in this section centered around equity price and price reaction following no voluntary disclosure. We next proceed to examine whether the presence of a mandatory disclosure requirement promotes or discourages voluntary disclosure probability by comparing this regime with a mandatory disclosure requirement to the one with no mandatory disclosure requirement. Figures 6 (a) and 6 (b) illustrate the role of mandatory disclosure requirement in influencing subsequent voluntary disclosure probability. A comparison of figures 6 (a) and 6 (b) reveals three interesting features. First, when prior default probability is less than 0.5, figure 6 (a) below illustrates that posterior voluntary disclosure probability in the presence of mandatory disclosures is mostly lower than such discretionary disclosure probability in the absence of such mandatory requirement, except for a small neighborhood around the maximum point of posterior voluntary disclosure probability in the presence of the mandatory disclosure requirement.
Second, when prior default probability is greater than 0.5, figure 6 (b) illustrates that posterior voluntary disclosure probability in the presence of a mandatory disclosure requirement is mostly greater than such discretionary disclosure probability in the absence of such mandatory disclosures, except for a small neighborhood around the minimum point of posterior voluntary disclosure probability in the presence of the mandatory disclosure requirement. Finally, in the absence of a mandatory disclosure requirement, the voluntary disclosure probability is significantly lower when the prior default probability is greater than 0.5 than when the prior default probability is less than 0.5. Thus, a comparison of figures 6 (a) and 6 (b) suggests that the answer to the question whether a mandatory disclosure requirement encourages or discourages voluntary disclosures depends on whether prior default probability is less than 0.5. If so, a mandatory disclosure requirement actually decreases the disclosure likelihood for sufficiently large and small values of public news.

5.2 Effect of analyst forecast consensus and dispersion on voluntary disclosures and equity prices

Extend the previous model to the case where the public information is an $n$-dimensional vector of analyst forecasts, for $n \geq 2$. Suppose that there are $n$ symmetric analysts, indexed by $i$, each of whom receives a private signal of the form $S_i = X + E_i$, where $E_i$ is a zero-
mean error term. We assume that \((X, E_1, \ldots, E_n)\) are uncorrelated and jointly distributed according to a multivariate \(t\) distribution with \(k\) degrees of freedom. The variance of \(X\) is \(\Sigma_X\) and the variance of \(E_i\) is \(\Sigma_E\). After observing his private signal, analyst \(i\) issues an unbiased forecast, \(Y_i\), of the firm’s liquidation value \(X\). Exploiting the properties of the elliptical distributions, the forecast is given by

\[
Y_i = \mu_X + \frac{\Sigma_X}{\Sigma_X + \Sigma_E} (S_i - \mu_X).
\]

The vector of forecasts \(Y \equiv (Y_1, \ldots, Y_n)\) is informationally equivalent to the vector of private signals \(S \equiv (S_1, \ldots, S_n)\). Therefore, to find the conditional distribution of \(X|S\) we can start from the joint distribution of \((X,Y)\). Letting \(t_n\) be an \(n\)-dimensional vector of ones and \(I_n\) be the \(n\)-dimensional identity matrix, we have

\[
\left( \begin{array}{c} X \\ Y \end{array} \right) \sim t_{n+1} \left( \begin{array}{c} \mu_X \\ \mu_X t_n \end{array} \right), \left( \begin{array}{cc} \Sigma_X & \Sigma_{XY} \\ \Sigma_{XY}' & \Sigma_Y \end{array} \right), k\right),
\]

where

\[
\Sigma_{XY} = \frac{\Sigma_X^2}{\Sigma_X + \Sigma_E},
\Sigma_Y = \frac{\Sigma_X}{\Sigma_X + \Sigma_E}^2 (\Sigma_E I_n + \Sigma_X t_n t_n').
\]

With this notation, the mean and variance of cash flow \(X\) conditional on the set of analysts’ reports \(Y\) are given by

\[
\begin{align*}
\mu(Y) &= \mu_X + \frac{n (\Sigma_X + \Sigma_E)}{\Sigma_E + n \Sigma_X} (C - \mu_X), \\
\sigma^2(Y) &= \frac{k}{k + n \Sigma_X} \frac{\Sigma_E \Sigma_X}{\Sigma_E + n \Sigma_X} + \frac{n}{k + n \Sigma_X} \left( \frac{\Sigma_E + \Sigma_X}{\Sigma_E + n \Sigma_X} \right)^2 \left[ D + \frac{\Sigma_E (C - \mu_Y)^2}{\Sigma_E + n \Sigma_X} \right],
\end{align*}
\]

where

\[
C \equiv \frac{\sum_{i=1}^n Y_i}{n} \text{ and } D \equiv \frac{\sum_{i=1}^n (Y_i - C)^2}{n}
\]

are analyst forecast consensus \((C)\) and forecast dispersion \((D)\), respectively, following Barron et al. (1998, 1999).

The next Proposition now predicts the firm’s posterior voluntary disclosure probability as a function of analysts’ forecast consensus \(C\) and dispensor \(D\).

\footnote{For a derivation, see the proof of Proposition 7.}
Proposition 7  The posterior probability of voluntary disclosure is:

(a) U-shaped in consensus, $C$, if the prior default probability is greater than 0.5;
(b) hump-shaped in consensus, $C$, if the prior default probability is less than 0.5; and
(c) decreasing (increasing) in forecast dispersion, $D$, if the posterior default probability is less (greater) than 0.5.

Propositions 7(a) and 7(b) results follow for similar reasons as in Propositions 4(a) and 4(b), respectively. Further, Proposition 7(c) result follows for similar reasons as in Proposition 3(b).

While the use of improper priors has several severe limitations, it is interesting to note that the next Corollary generates predictions that are similar to the one with a conjugate distribution whose posterior precision does not vary in the realized value of the public signal.

Corollary 3  When the prior on the liquidation value $X$ is improper (i.e., as $\Sigma_X \to \infty$), the posterior probability of voluntary disclosure is:

(a) increasing everywhere in consensus $C$; and
(b) increasing (decreasing) in the precision of analysts’ information, $\Sigma^{-1}_E$, if the posterior default probability is less (greater) than 0.5.

Figures 7 (a) and 7 (b) below depict how posterior disclosure probability varies in analysts’ forecast consensus and dispersion, respectively.
It is interesting to note from figure 7(a) that for sufficiently low and high values of analyst consensus values, posterior disclosure probability is greater when prior default probability is less than 0.5 than when more than 0.5. Only for moderate values of consensus around the prior mean of cash flows we get a higher posterior disclosure probability with a prior default probability being greater than 0.5. In other words, when prior beliefs about default probability are low, it takes a sufficiently strong deviation of consensus number away from the prior mean of cash flow to induce the manager to disclose more often; otherwise, the manager’s nondisclosure elicits relatively less market skepticism, thereby reducing their voluntary disclosure incentives. Figure 7(b) illustrates two results: (i) first, that the posterior disclosure probability is greater for every dispersion value when posterior default probability is less than 0.5 than when greater than 0.5; and (ii) that the posterior disclosure probability decreases (increases) in dispersion value $D$ when posterior default probability is less (greater) than 0.5. The intuition for the first result (i) from figure 7(a) is straightforward and directly follows from applying Proposition 1 result that the posterior disclosure probability is inversely related to the posterior default probability. The intuition for the second result (ii) from figure 7(b) is a bit more involved and follows next. When the posterior default probability is less than 0.5, the numerator in the posterior default probability $G \left( \frac{\delta - \mu_x}{\sigma(Y)} \right)$ is negative and the denominator increases in the dispersion value $D$, thus leading to a lower
negative fraction. Consequently, the CDF \( G(\cdot) \) increases in dispersion \( D \), thus increasing the posterior default probability, which feature in turn, leads to a lower posterior disclosure probability from applying the result in Proposition 1. In contrast, when the posterior default probability is greater than 0.5, the numerator in the posterior default probability \( G \left( \frac{\mu - \bar{x}}{\sigma(Y)} \right) \) is positive. So, as the denominator increases in the dispersion value \( D \), the CDF \( G(\cdot) \) decreases in dispersion \( D \), thus decreasing the posterior default probability. Hence, from Proposition 1 it follows that the posterior disclosure probability increases in dispersion \( D \) with a high enough posterior default probability. It must be noted that figure 7 (b) illustrates the effect of a change in dispersion, holding a given consensus level \( C \) fixed. Further, it must be pointed out that figure 7 (b) is the only setting in which the comparative statics representation varies in posterior (and not prior) default probability.

6 Conclusion

We are currently working on several extensions of this setting including examining the behavior of debt prices, design of debt contracts, managers' different objective functions, cost of capital, dynamic disclosures and other related topics. In fact, a companion paper examines the firm’s disclosure dynamics in the presence of possible loan sales by the lender in a secondary market.

Appendix

Proof of Proposition 1. Using integrals, Equation (2) can be rewritten as

\[
\hat{x} - \delta = \frac{(1 - q) \int_{\delta}^{\infty} (x - \delta) f_{X|Y=y} (x) \, dx + q \int_{\delta}^{\hat{x}} (x - \delta) f_{X|Y=y} (x) \, dx}{(1 - q) + qF(\hat{x})}.
\]

Next, note that \( P [X < x|Y] = G \left( \frac{x-\mu}{\sigma} \right) \), where \( G(\cdot) \) is the cumulative function of \( U \), and so

\[
f_{X|Y=y} (x) = \frac{1}{\sigma} g \left( \frac{x-\mu}{\sigma} \right), \quad \text{where } g(\cdot) \text{ is the density function of } U.
\]

By applying the change of variable \( x = \mu + \sigma u \) to the right-hand side, we obtain

\[
\hat{x} - \delta = \frac{(1 - q) \int_{\frac{\mu - \delta}{\sigma}}^{\infty} (\mu + \sigma u - \delta) g(u) \, du + q \int_{\frac{\mu - \delta}{\sigma}}^{\frac{\mu - \hat{x}}{\sigma}} (\mu + \sigma u - \delta) g(u) \, du}{(1 - q) + qG \left( \frac{\mu - \hat{x}}{\sigma} \right)}.
\]
Further rearranging, we have
\[
\tilde{x} - \tilde{\delta} = \frac{(1 - q) \left[ \int_{-\infty}^{\tilde{\delta}} (u - \tilde{\delta}) g(u) \, du - \int_{-\infty}^{\tilde{x}} (u - \tilde{x}) g(u) \, du \right] + q \int_{\tilde{\delta}}^{\tilde{x}} (u - \tilde{\delta}) g(u) \, du}{(1 - q) + qG(\tilde{x})},
\]
where
\[
\tilde{x} \equiv \frac{\tilde{x} - \mu}{\sigma}, \quad \tilde{\delta} \equiv \frac{\delta - \mu}{\sigma}
\]
are the “standardized” disclosure threshold and face value of the debt. Using integration by parts, the indifference condition becomes
\[
\tilde{x} - \tilde{\delta} = \frac{(1 - q) \left[ -\tilde{\delta} + \int_{-\infty}^{\tilde{\delta}} G(u) \, du \right] + q \left[ \left( \tilde{x} - \tilde{\delta} \right) G(\tilde{x}) - \int_{\tilde{\delta}}^{\tilde{x}} G(u) \, du \right]}{(1 - q) + qG(\tilde{x})},
\]
and thus
\[
(1 - q) \left[ \tilde{x} - \int_{-\infty}^{\tilde{\delta}} G(u) \, du \right] + q \int_{\tilde{\delta}}^{\tilde{x}} G(u) \, du = 0. \tag{8}
\]

The posterior disclosure probability is \( q \left[ 1 - G(\tilde{x}) \right] \), hence it is decreasing in the equilibrium standardized disclosure threshold \( \tilde{x} \). The posterior default probability is \( G(\tilde{\delta}) \), which is increasing in the standardized face value \( \tilde{\delta} \). We then show that the equilibrium \( \tilde{x} \) is an increasing function of \( \tilde{\delta} \). To this purpose, totally differentiate (8) with respect to \( y \),
\[
(1 - q) \left[ \tilde{x}' - G(\tilde{\delta}) \tilde{\delta}' \right] + q \left[ G(\tilde{x}) \tilde{x}' - G(\tilde{\delta}) \tilde{\delta}' \right] = 0,
\]
whence
\[
\tilde{x}' = \frac{G(\tilde{\delta})}{(1 - q) + qG(\tilde{x})} \tilde{\delta}'. \tag{9}
\]
Since \( \text{sign} (\tilde{x}') = \text{sign} (\tilde{\delta}') \), the posterior disclosure probability is higher when \( \tilde{\delta} \) is lower. ■

**Proof of Remark 1.** An increase in the amount of borrowing \( M \) is mathematically equivalent to an increase in \( \delta \). As \( \delta \) increases, \( \tilde{\delta} \) increases and, by Proposition 1, the disclosure probability decreases. ■

**Proof of Remark 2.** Suppose that \((X, Y)\) follow a joint multivariate \( t \) distribution with \( k \) degrees of freedom, and that \( Y \) and \( X \) are \( n_Y \)-dimensional and one-dimensional random vectors, respectively. Define
\[
\mu(y) \equiv \mu_X - \Sigma_{XY} \Sigma_Y^{-1} (y - \mu_Y),
\]
\[
\sigma^2(y) \equiv (\Sigma_X - \Sigma_{XY} \Sigma_Y^{-1} \Sigma_X') \left( \frac{k + (y - \mu_Y)' \Sigma_Y^{-1} (y - \mu_Y)}{k + n_Y} \right).
\]
Then,

$$X \mid (Y = y) \sim t_{n_X} (\mu (y), \sigma^2 (y), k + n_Y),$$

that is, the posterior distribution of $X$ is also multivariate $t$ (e.g., Ding (2016)).

If we let $U_{k+n_Y}$ denote a standard univariate $t$ distribution with $k+n_Y$ degrees of freedom, by the properties of affine transformation of elliptically distributed random variables (e.g., Gomez et al. (2003)),

$$\mu (y) + \sigma (y) U_{k+n_Y} \sim t_{n_X} (\mu (y), \sigma^2 (y), k + n_Y),$$

and therefore $X \mid (Y = y)$ is equal in distribution to $\mu (y) + \sigma (y) U_{k+n_Y}$. ■

**Proof of Proposition 2.** We first derive the expression in (4). Let $f_{X \mid Z=z} (\cdot)$ and $F_{X \mid Z=z} (\cdot)$ denote the posterior density and cumulative functions for $X$. The posterior equity value is

$$E [V_E (X) \mid Z = z] = \int_{\delta}^{\infty} (x-\delta) f_{X \mid Z=z} (x) dx = \int_{\delta}^{\infty} (x-\delta) \frac{1}{\sigma} g_k \left( \frac{x-\mu}{\sigma} \right) dx$$

$$= \sigma \int_{\delta}^{\infty} \left( u-\tilde{\delta} \right) g_k (u) du,$$

where: the first equality follows from the definition; the second equality from invoking the fact that $F_{X \mid Z=z} (x) = G_k \left( \frac{x-\mu}{\sigma} \right)$ and taking the derivative of $G_k \left( \frac{x-\mu}{\sigma} \right)$ with respect to $x$; the third from operating the change of variable $u = \frac{x-\mu}{\sigma}$.

**Proof of Part (a).** Note that $k \to \infty$, $\mu' (z) > 0$ but $\sigma' (z) = 0$. We begin by showing that the default probability is decreasing in the realized news $z$, namely, that $\tilde{\delta}' (z) < 0$. This follow from $\tilde{\delta}' = -\frac{\mu'}{\sigma} < 0$. Next, we show that $E [V_E (X) \mid Z = z]$ is increasing in $z$:

$$\frac{\partial E [V_E (X) \mid Z = z]}{\partial z} = \sigma' \int_{\delta}^{\infty} \left( u-\tilde{\delta} \right) g_k (u) du + \sigma \left[ - \tilde{\delta}' g_k (u) - \int_{\delta}^{\infty} g_k (u) du \right] \tilde{\delta}'$$

$$= -\sigma \left[ 1 - G_k \left( \tilde{\delta} \right) \right] \tilde{\delta}' > 0.$$

**Proof of Part (b.i).** From the proof of Proposition 4 we know that when the prior default probability is lower than 0.5 we have $\tilde{\delta}' (z) < 0$ for $z < z^*$ and $\tilde{\delta}' (z) > z^*$. From the proof of Proposition 2 we also know that $\lim_{z \to -\infty} \tilde{\delta} (z) = l > 0$ and $\lim_{z \to -\infty} \tilde{\delta} (z) = -l < 0$, where $l < \infty$. Taking the derivative of $\sigma (z)$ with respect to the realized news $z$ gives

$$\sigma' (z) = \sigma_X \sqrt{1 - \rho_X Y} \frac{z}{k+1} \frac{\sqrt{k+z^2}}{\sqrt{k+z^2}},$$

and therefore: $\sigma' (z) < 0$ for $z < 0$ and $\sigma' (z) > 0$ for $z > 0$; $\lim_{z \to -\infty} \sigma' (z) = -\rho_X Y \sigma_X / l < 0$
and \( \lim_{z \to \infty} \sigma'(z) = \rho_X \sigma_X / l > 0 \). Also, consider the product

\[
\sigma(z) \tilde{\delta}'(z) = \left[ \sigma_X \sqrt{1 - \rho_{XY}^2} \frac{k + 1}{k + 1} \sqrt{k + z^2} \right] \left[ -\rho_{XY} \sigma_X \frac{k + \sqrt{k + z^2}}{\sqrt{k + z^2}} \frac{(\delta - \mu_X - \rho_{XY} \sigma_X z)}{\sqrt{k + z^2}} \right] \\
= -\rho_{XY} \sigma_X \frac{k - (\delta - \mu_X) z}{(k + z^2)},
\]

hence \( \lim_{z \to -\infty} \sigma(z) \tilde{\delta}'(z) = \lim_{z \to \infty} \sigma(z) \tilde{\delta}'(z) = 0 \).

The derivative of the posterior equity value is

\[
\frac{\partial E[V_E(X) \mid Z = z]}{\partial z} = \sigma' \int_{\tilde{\delta}}^{\infty} (u - \tilde{\delta}) g_k(u) du - \sigma \tilde{\delta}' \left[ 1 - G_k(\tilde{\delta}) \right],
\]

and so, using the limit properties above,

\[
\lim_{z \to -\infty} \frac{\partial E[V_E(X) \mid Z = z]}{\partial z} = -\rho_{XY} \sigma_X \frac{l}{l} \int_{-l}^{\infty} (u - l) g_k(u) du < 0
\]

\[
\lim_{z \to \infty} \frac{\partial E[V_E(X) \mid Z = z]}{\partial z} = \rho_{XY} \sigma_X \frac{l}{l} \int_{l}^{\infty} (u + l) g_k(u) du > 0.
\]

Thus, in these limits the derivative of the posterior equity value has the same sign as \( \tilde{\delta}'(z) \).

**Proof of Part (b.ii).** From the proof of Proposition 4 we know that when the prior default probability is greater than 0.5 we have \( \tilde{\delta}'(z) > 0 \) for \( z < z^{**} \) and \( \tilde{\delta}'(z) < z^{**} \) for \( z > z^{**} \), where the cutoff \( z^{**} < 0 \). Therefore, \( \sigma'(z^{**}) < 0 \) and \( \tilde{\delta}'(z^{**}) = 0 \) imply (by continuity) that \( E[V_E(X) \mid Z = z] \) is decreasing in a neighborhood to the right of \( z^{**} \), where the default probability is also decreasing.

**Proof of Part (b.iii).** The claim follows from Lemma 1. ■

**Proof of Proposition 3.** To prove the results, we take the derivative of \( \tilde{\delta} \) with respect to the various parameters.

**Proof of Part (a).** By inspection of \( \tilde{\delta} \).

**Proof of Part (b).** We have \( \tilde{\delta}'(\sigma_X) \propto (\delta - \mu_X) \).

**Proof of Part (c).** Differentiation with respect to \( \rho_{XY} \) yields

\[
\tilde{\delta}'(\rho_{XY}) \propto -\sigma_X z \left( 1 - \rho_{XY}^2 \right) + (\delta - \mu_X - \rho_{XY} \sigma_X z) \frac{\rho_{XY}}{\sqrt{1 - \rho_{XY}^2}} \\
\propto (\delta - \mu_X) \rho_{XY} - \sigma_X z.
\]

For \( z \to -\infty \), \( \tilde{\delta}'(\rho_{XY}) \) is positive, whereas for \( z \to \infty \), \( \tilde{\delta}'(\rho_{XY}) \) is negative. ■

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Proof of Proposition 4. The prior default probability is \( P[X < \delta] = G_k\left(\frac{\delta - \mu_X}{\sqrt{\Sigma_X}}\right) \), where \( G_k(\cdot) \) is the cumulative function of the standard univariate \( t \) with \( k \) degrees of freedom. This claim follows from the fact that the marginal on \( X \) of the joint distribution of \((X, Y)\) is the univariate \( t_1(\mu_X, \Sigma_X, k) \). Since the \( t \) distribution is symmetric around its mean, \( P[X < \delta] > 0.5 \) if and only if \( \delta - \mu_X > 0 \).

Proof of Part (a). Take the derivative of \( \tilde{\delta} \) with respect to \( z \),

\[
\tilde{\delta}'(z) = \frac{-\rho_{XY} \sigma_X \sqrt{\frac{k+2}{k+1}} - \frac{(\delta - \mu_X - \rho_{XY} \sigma_X z) z}{\sqrt{\frac{k+2}{k+1}}}}{\sigma_X \sqrt{(1 - \rho_{XY}^2) \frac{k+2}{k+1}}},
\]

so \( \tilde{\delta}'(z) > 0 \) if and only if \( (\delta - \mu_X) z < -\rho_{XY} \sigma_X k \). When \( \delta - \mu_X < 0 \), \( \tilde{\delta}'(z) > 0 \) if and only if \( z > z^* \equiv \frac{\rho_{XY} \sigma_X k}{\delta - \mu_X} \). Therefore, for realizations \( z < z^* \) \((z > z^*)\) the probability of disclosure is increasing \((\text{decreasing})\) in \( z \), and maximized at \( z = z^* \).

Proof of Part (b). When \( \delta - \mu_X > 0 \), \( \tilde{\delta}'(z) > 0 \) if and only if \( z < z^{**} \equiv -\frac{\rho_{XY} \sigma_X k}{\delta - \mu_X} \). Hence, for realizations \( z < z^{**} \) \((z > z^{**})\) the probability of disclosure is decreasing \((\text{increasing})\) in \( z \), and minimized at \( z = z^{**} \).

Proof of Corollary 1. When \( \delta - \mu_X = 0 \), \( \tilde{\delta}'(z) \propto -\rho_{XY} \sigma_X k < 0 \), and so the disclosure probability is always increasing in \( z \).

Proof of Corollary 2. By Proposition 1, the posterior disclosure probability is a decreasing function of the posterior default probability. Thus, we study the limits of the default probability as \( z \to -\infty \) and \( z \to \infty \), and show that the default probability is lower in the latter limit.

Given Assumption 2 on a joint \( t \) distribution, the posterior default probability is

\[
P[X < \delta | Z = z] = P[\mu(Z) + \sigma(Z) U_{k+1} < \delta | Z = z] = G_{k+1}\left(\tilde{\delta}\right),
\]

where

\[
\tilde{\delta} \equiv \frac{\delta - \mu_X - \rho_{XY} \sigma_X z}{\sigma_X \sqrt{(1 - \rho_{XY}^2) \frac{k+2}{k+1}}},
\]

and \( G_{k+1}(\cdot) \) is the cumulative function of a standard univariate \( t \) distribution with \( k + 1 \) degrees of freedom. The claim follows because

\[
\lim_{z \to -\infty} \tilde{\delta} = l > \lim_{z \to \infty} \tilde{\delta} = -l,
\]

where \( l \equiv \rho_{XY} / \sqrt{(1 - \rho_{XY}^2) (k + 1)^{-1}} > 0 \).
Proof of Proposition 5. First, note that \( P(\{nd, z\}) = \sigma(z - \tilde{\delta}) \). Hence, taking the derivative with respect to the news \( z \) or any parameter,

\[
P'(\{nd, z\}) = \sigma' \left( \tilde{x} - \tilde{\delta} \right) + \sigma \left( \tilde{x}' - \tilde{\delta}' \right)
\]

\[
= \sigma' \left( \tilde{x} - \tilde{\delta} \right) - \tilde{\delta}' \sigma \left[ 1 - \frac{G(\tilde{\delta})}{(1 - q) + qG(\tilde{x})} \right]
\]

\[
= \sigma' \left[ \tilde{x} - \frac{G(\tilde{\delta})}{(1 - q) + qG(\tilde{x})} \tilde{\delta} \right] + \left[ 1 - \frac{G(\tilde{\delta})}{(1 - q) + qG(\tilde{x})} \right] \mu',
\]

where the second equality follows from (9) and the last equality from \( \tilde{\delta}' = \frac{1}{\sigma} (-\mu' - \sigma' \tilde{\delta}) \).

Proof of Part (a). \( \sigma'(\mu_X) = 0 \) and \(-\tilde{\delta}'(\mu_X) > 0 \) (by Proposition 3(a)).

Proof of Part (b). \( \sigma'(\rho_{XY}) < 0 \) and \(-\tilde{\delta}'(\rho_{XY}) < 0 \) for \( z \) sufficiently low (by Proposition 3(d)).

Proof of Part (c). Note that, in equilibrium, it holds that \( \tilde{x} > \tilde{\delta} \). This inequality implies \( G(\tilde{\delta}) < (1 - q) + qG(\tilde{x}) \). Thus, when \( \sigma' \) and \( \tilde{\delta}' \) have opposite signs, the sign of \( P'(\{nd, z\}) \) is equal to the sign of \( \sigma' \). The posterior volatility is U-shaped \( z \) and achieves its minimum at \( z = 0 \). When the prior default probability is less than 0.5, \( \tilde{\delta}' < 0 \) to the left of \( z^* \) (by Proposition 4(a)). Since \( z^* > 0 \), for \( z \in (0, z^*) \) \( \text{sign}(\sigma') = -\text{sign}(\tilde{\delta}') > 0 \), which, by our previous consideration, implies \( P'(\{nd, z\}) = \text{sign}(\sigma') > 0 \) in this interval.

Proof of Part (d). When the prior default probability is greater than 0.5, \( \tilde{\delta}' > 0 \) to the right of \( z^* \) (by Proposition 4(b)). In particular, \( \tilde{\delta}' > 0 \) for \( z > 0 \), since \( z'^* < 0 \). For \( z < z^* \), \( \text{sign}(\sigma') = -\text{sign}(\tilde{\delta}') < 0 \), and so \( P'(\{nd, z\}) = \text{sign}(\sigma') < 0 \). Vice versa, for \( z > 0 \), \( \text{sign}(\sigma') = -\text{sign}(\tilde{\delta}') > 0 \), and so \( P'(\{nd, z\}) = \text{sign}(\sigma') > 0 \).

Proof of Part (e). \( \sigma'(\sigma_X) > 0 \) and \(-\tilde{\delta}'(\sigma_X) > 0 \) when the prior default probability is greater than 0.5 (by Proposition 3(b)).

Further, \( \sigma'(\sigma_X) > 0 \) and \(-\tilde{\delta}'(\sigma_X) = 0 \) when the prior default probability is 0.5 (by Proposition 3(c)).

Proof of Part (f). When the prior default probability is equal to 0.5, \( \tilde{\delta}' < 0 \) for all \( z \) (by Proposition 4(c)). Therefore, for \( z > 0 \) we have \( \text{sign}(\sigma') = -\text{sign}(\tilde{\delta}') > 0 \), and so \( P'(\{nd, z\}) = \text{sign}(\sigma') > 0 \).

Proof of Proposition 6. Note that \( P(\{nd, z\}) = \sigma \left( \tilde{x} - \tilde{\delta} \right) \) and \( P(z) = \sigma \int_{\tilde{\delta}}^{\infty} \left( u - \tilde{\delta} \right) g_k(u) \, du \).
Using integration by parts, the posterior equity price can be written as

\[ P(z) = \sigma \left[ \int_{-\infty}^{\infty} (u - \bar{\delta}) g_k(u) \, du - \int_{-\infty}^{\bar{\delta}} (u - \bar{\delta}) g_k(u) \, du \right] = \sigma \left[ -\bar{\delta} + \int_{-\infty}^{\bar{\delta}} G_k(u) \, du \right]. \]

Therefore,

\[ \frac{P_{2E}(\{z, nd\}) - P_{1E}(z)}{\sigma(z)} = \bar{\xi} - \int_{-\infty}^{\bar{\delta}} G_k(u) \, du. \]

Its derivative is

\[ \bar{\xi}' - G_k(\bar{\delta}) \bar{\delta}' = \delta' G_k(\bar{\delta}) \left[ \frac{1}{(1 - q) + qG_k(\bar{x})} - 1 \right]. \]

Because \((1 - q) + qG_k(\bar{x})\), the sign of the scaled price reaction is the same as the sign of \(\bar{\delta}'\).

The rest of the proof follows along prior lines. ■

**Proof of Proposition 7.** We first derive Equations (6) and (7). Let \(\Pi_n \equiv \ell_n (\ell_n^t \ell_n)^{-1} \ell_n^t\). Then,

\[ \Sigma_Y = \left( \frac{X}{X + \Sigma} \right)^2 [\Sigma (\Pi_n + I_n - \Pi_n) + n\Sigma X \Pi_n] = \left( \frac{X}{X + \Sigma} \right)^2 [(\Sigma + n\Sigma X) \Pi_n + \Sigma (I_n - \Pi_n)], \]

hence

\[ \Sigma_Y^{-1} = \left( \frac{X}{X + \Sigma} \right)^{-2} \left[ \frac{1}{\Sigma + n\Sigma X} \Pi_n + \frac{1}{\Sigma} (I_n - \Pi_n) \right]. \]

To compute the posterior mean, observe that

\[ \mu(Y) = \mu_X + \Sigma_{XY} \Sigma_Y^{-1} (Y - \mu_X \ell_n) \]

\[ = \mu_X + (\Sigma_X + \Sigma_E) \ell_n^t \left[ \frac{1}{\Sigma + n\Sigma X} \Pi_n + \frac{1}{\Sigma} (I_n - \Pi_n) \right] (Y - \mu_X \ell_n) \]

\[ = \mu_X + (\Sigma_X + \Sigma_E) \ell_n^t \frac{(Y - \mu_X \ell_n)}{\Sigma + n\Sigma X} = \mu_X + (\Sigma_X + \Sigma_E) \frac{n(C - \mu_X)}{\Sigma + n\Sigma X}, \]

\[ \text{11} \text{One sees that this expression corresponds to the inverse of } \Sigma_Y \text{ by checking that } \Sigma_Y \Sigma_Y^{-1} = I_n. \text{ The latter identity is due to the following properties of the projection matrix: } \Pi_n \text{ is idempotent (hence } \Pi_n \Pi_n = \Pi_n); \text{ and } \ell_n^t (I_n - \Pi_n) = 0 \text{ implies } \Pi_n (I_n - \Pi_n) = 0. \]
which has exploited the fact that \( 
u_n (I_n - \Pi_n) = 0 \). As to the posterior variance, note that

\[
\Sigma_X - \Sigma_{XY} \Sigma_Y^{-1} \Sigma_{XY}' = \frac{\Sigma_E \Sigma_X}{\Sigma_E + n \Sigma_X},
\]

and

\[
(Y - \mu_Y t)' \left[ \frac{1}{\Sigma_E + n \Sigma_X} \Pi_n + \frac{1}{\Sigma_E} (I_n - \Pi_n) \right] (Y - \mu_Y t) = \left\{ Y' \left[ \frac{1}{\Sigma_E + n \Sigma_X} \Pi_n + \frac{1}{\Sigma_E} (I_n - \Pi_n) \right] - \mu_Y \frac{1}{\Sigma_E + n \Sigma_X} \right\} (Y - \mu_Y t)
\]

\[
= \frac{1}{\Sigma_E + n \Sigma_X} Y' \Pi_n Y + \frac{1}{\Sigma_E} Y' (I_n - \Pi_n) Y - 2 \frac{\mu_Y}{\Sigma_E + n \Sigma_X} Y' t + \frac{n \mu_Y^2}{\Sigma_E + n \Sigma_X}
\]

\[
= \frac{n C^2}{\Sigma_E + n \Sigma_X} + \frac{n D}{\Sigma_E} - 2 \frac{\mu_Y n C}{\Sigma_E + n \Sigma_X} + \frac{n \mu_Y^2}{\Sigma_E + n \Sigma_X} = \frac{D}{\Sigma_E} + \frac{(C - \mu_Y)^2}{\Sigma_E + n \Sigma_X}.
\]

Hence,

\[
(Y - \mu_Y t)' \Sigma_Y^{-1} (Y - \mu_Y t) = \left( \frac{\Sigma_E + \Sigma_X}{\Sigma_X} \right)^2 n \left[ \frac{D}{\Sigma_E} + \frac{(C - \mu_Y)^2}{\Sigma_E + n \Sigma_X} \right]
\]

and

\[
\sigma^2(Y) = \frac{\Sigma_E \Sigma_X}{\Sigma_E + n \Sigma_X} \left( \frac{\Sigma_E + \Sigma_X}{\Sigma_X} \right)^2 \frac{k + n}{k + n} \left[ \frac{D}{\Sigma_E} + \frac{(C - \mu_Y)^2}{\Sigma_E + n \Sigma_X} \right]
\]

\[
= \frac{\Sigma_E \Sigma_X}{\Sigma_E + n \Sigma_X} \left( \frac{\Sigma_E + \Sigma_X}{\Sigma_X} \right)^2 \frac{k + n}{k + n} \left[ \frac{D}{\Sigma_E} + \frac{(C - \mu_Y)^2}{\Sigma_E + n \Sigma_X} \right]
\]

\[
= \frac{\Sigma_E \Sigma_X}{\Sigma_E + n \Sigma_X} k + \frac{n (\Sigma_E + \Sigma_X)^2}{\Sigma_X (\Sigma_E + n \Sigma_X)} \left[ D + \frac{\Sigma_E (C - \mu_Y)^2}{\Sigma_E + n \Sigma_X} \right].
\]

\[
\lim_{\Sigma_X \to \infty} \Sigma_Y^{-1} = \frac{1}{\Sigma_E} (I_n - \Pi_n),
\]

we have

\[
\lim_{\Sigma_X \to \infty} (Y - \mu_Y t_n)' \Sigma_Y^{-1} (Y - \mu_Y t_n) = \Sigma_E^{-1} (Y - \mu_Y t_n)' (I_n - \Pi_n) (Y - \mu_Y t_n)
\]

\[
= \Sigma_E^{-1} [Y' (I_n - \Pi_n) - \mu_Y t_n' (I_n - \Pi_n)] (Y - \mu_Y t_n)
\]

\[
= \Sigma_E^{-1} Y' (I_n - \Pi_n) Y = \Sigma_E^{-1} n D.
\]

Proof of Parts (a and b). By inspection of \( \tilde{\delta} \) we see that the default probability decreases
if and only if $\mu'(C) \sigma + \sigma'(C) (\delta - \mu) > 0$. Observe that
\[
\mu'(C) = \frac{n (\Sigma_X + \Sigma_E)}{\Sigma_E + n \Sigma_X},
\]
\[
\sigma'(C) = \frac{n \frac{\Sigma_E (\Sigma_E + \Sigma_X)^2}{k + n \Sigma_X (\Sigma_E + n \Sigma_X)^2} (C - \mu_Y)}{\sigma}.
\]

It follows that $\delta'(C) < 0$ if and only if
\[
0 < \frac{n (\Sigma_X + \Sigma_E)}{\Sigma_E + n \Sigma_X} \left\{ \frac{k}{k + n} \frac{\Sigma_E \Sigma_X}{\Sigma_E + n \Sigma_X} + \frac{n (\Sigma_E + \Sigma_X)^2}{k + n \Sigma_X (\Sigma_E + n \Sigma_X)} \left[ D + \frac{\Sigma_E (C - \mu_Y)^2}{\Sigma_E + n \Sigma_X} \right] \right\}
+ \frac{n \frac{\Sigma_E (\Sigma_E + \Sigma_X)^2}{k + n \Sigma_X (\Sigma_E + n \Sigma_X)^2} (C - \mu_Y)}{\sigma} \left[ \delta - \mu_X - \frac{n (\Sigma_X + \Sigma_E)}{\Sigma_E + n \Sigma_X} (C - \mu_X) \right],
\]
that is, if and only if
\[
0 < \left\{ \frac{k}{k + n} \frac{\Sigma_X^2}{\Sigma_E + \Sigma_X} + \frac{\Sigma_E + \Sigma_X}{\Sigma_E} D \right\} + (\delta - \mu_X) (C - \mu_Y).
\]
The claims follow from inspection of (10).

**Proof of Part (c).** The posterior default probability is less than 0.5 if and only if its numerator, $\delta - \mu (Y)$, is negative. In that case, increasing $D$ at the denominator leads to an increase in $\tilde{\delta}$ (namely, $\tilde{\delta}$ becomes less negative). Therefore, if $\delta - \mu (Y) < 0$ the default probability is higher when the dispersion is higher. As a consequence, the disclosure probability is lower when $\delta - \mu (Y) < 0$ and $D$ is greater. Oppositely, the disclosure probability is greater when $\delta - \mu (Y) > 0$ and $D$ is lower, since this combination corresponds to a lower probability of default. When $\delta - \mu (Y) = 0$, changes in the denominator do not affect the posterior default probability.

**Proof of Corollary 3.** When $\Sigma_X \to \infty$, the mean and variance of the liquidation value conditional on the analysts’ reports are given by
\[
\mu (Y) = C \text{ and } \sigma^2(Y) = \frac{\Sigma_E \frac{k}{n} + D}{k + n}.
\]
The claims follow from inspection of $\tilde{\delta}$ evaluated using the expressions in (11).
A References


