Optimal Investment in Credit Derivatives Portfolio under Contagion Risk

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Abstract

We consider the optimal portfolio problem of a power investor who wishes to allocate her wealth between several credit default swaps (CDSs), a stock index, and a money market account. We model contagion risk among the reference entities in the portfolio using a reduced form Markovian model with interacting default intensities. Using the dynamic programming principle, we establish a lattice dependence structure between the Hamiltonian-Jacobi-Bellman equations associated with the default states of the portfolio. We show existence and uniqueness of a classical solution to each equation and characterize them in terms of solutions to inhomogeneous Bernoulli’s type ODEs. We perform a numerical analysis to assess the impact of default contagion and find that the increased intensity triggered by default of a very risky entity strongly impacts size and directionality of the investor strategy. Such findings outline the key role played by default contagion when investing in portfolios subject to multiple sources of default risk.

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1 Introduction

Since the seminal work of Merton (1969), who pioneered the use of optimal stochastic control techniques to solve continuous time portfolio optimization problems, there has been enormous interest in continuous time utility maximization problems, see Karatzas et al. (1996), and Fleming and Pang (2004). Most of the proposed models have dealt with markets consisting of default-free securities such as stocks, where the uncertainty in price is typically governed by a continuous process, chosen to be a Brownian motion. The economy-wide crisis initiated at the end of nineties in Latin American countries, as well as the global financial crisis of 2007-2008 resulting in the failure of systemically important financial institutions, have outlined the importance of including the default feature into the models. As a consequence, many researchers have started to analyze portfolio optimization problems in defaultable markets, as discussed next.

Korn and Kraft (2003) studied optimal portfolio problems with defaultable assets within a Merton structural default framework. Kraft and Steffensen (2005) extended the analysis to deal with a Black-Cox framework, where default is defined as the first passage time of an economic state process below a given threshold. A different branch of the literature has modeled default events using a reduced form approach. Starting from the seminal paper of Merton (1971) who assumes constant interest rate and default intensity, many other extensions have been proposed some of which are surveyed next. Bielecki and Jang (2006) derived optimal investment strategies for a CRRA investor, allocating her wealth among a defaultable bond, risk-free bank account, and a stock. Bo et al. (2010) considered an infinite horizon portfolio optimization problem, where a logarithmic investor can choose a consumption rate, and invest her wealth across a defaultable perpetual bond, a stock, and a money market account. Lakner and Liang (2008) employed duality theory to obtain the optimal investment strategy in a market consisting of a defaultable bond and a money market account under a continuous time model where bond prices can jump. Capponi and Lopez (2012) considered a Markov modulated economy driving the prices of stock and defaultable bond securities. Using the HJB approach, they

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recover the optimal investment strategies as the unique solution to a coupled system of partial differential
equations. Bielecki et al. (2008a) considered a Markov modulated default intensity framework for pricing and
hedging defaultable game options using forward-backward SDEs. Jiao and Pham (2011) combined duality
theory and dynamic programming to optimize the utility of a CRRA investor, in a market consisting of a
riskless bond and a stock subject to counterparty risk.

The literature surveyed above has considered markets consisting of one defaultable security. This prevents
the analysis of default contagion effects on optimal portfolio allocations, despite the fact that contagion risk
plays a fundamental role in financially distressed situations. As demonstrated during the crisis, contagion
phenomena may be responsible for high mark-to-market variations in prices of credit sensitive securities.
There are only few studies, discussed next, which have considered portfolio frameworks with multiple de-
faultable securities. Kraft and Steffensen (2008) considered an investor who can allocate her wealth across
multiple defaultable bonds, in a model with constant default intensity and risk premium, but where simul-
taneous defaults are allowed. In the same market model, Kraft and Steffensen (2009) defined default as the
beginning of financial distress, and discussed contagion effects on pricing of defaultable bonds. Callegaro et
al. (2010) considered an investor who can allocate her wealth across several defaultable assets, whose dis-
crete dynamics depends on a partially observed exogenous factor process. Most recently, Jiao et al. (2013)
analyzed a portfolio framework under multiple jumps and default events. Using the density hypothesis, they
can separate before and after default scenarios, and establish existence and uniqueness of the value function
via recursive systems of backward stochastic differential equations.

We next summarize our main contributions. Firstly, we consider a market model consisting of a stock
index and of credit derivatives, namely credit default swaps (CDSs). To the best of our knowledge, ours
represent the first attempt in the literature to include credit derivatives in a portfolio optimization framework.
Earlier works discussed above consider defaultable assets to be primary securities, such as corporate bonds.
From an empirical point of view, the continuous time framework is much better suited for CDSs, rather than
corporate bonds. As described in Currie and Morris (2002), credit has become much more tradable due to
the development of the credit default swap market, consequently trading strategies based both on equity
and CDSs have become quite popular.

Secondly, we develop our analysis within a contagion credit risk model with interacting default intensities.
In our model, the default of one name may trigger an increase in default intensities of other names in the
portfolio. This in turn leads to jumps in the market valuation of credit default swaps referencing the surviving
names, and consequently to jumps in the optimal wealth proportion allocated to these CDSs. We refer the
reader to Frey and Backhaus (2004) for a theoretical illustration, and to Jarrow and Yu (2001) for more
explicit examples of default contagion effects in interacting intensity models. The Markovian nature of the
intensity process leads to explicit expressions for the market value of each CDS in the portfolio recovered as
solutions to Feynman-Kac equations. The interacting intensity feature creates an explicit default contagion
term in the dynamics of each CDS market value (the precise statement is in Theorem 2.3). Clearly, when
there is only one name in portfolio the contagion effect disappears, and our framework reduces to Bielecki
et al. (2008b) (see Remark 2.5).

We consider a power investor who maximizes expected utility from terminal wealth in a finite horizon, and
characterize the optimal investment strategies using a HJB dynamic programming approach. The interacting
nature of the default intensity process leads to establishing a lattice dependence structure, where the partial
order is induced by the default state of the portfolio. As we demonstrate later in the paper, the value
function corresponding to the state with all names defaulting acts as the maximum, while the one associated
to the state where all names are alive acts as the minimum. Following the dynamic programming principle,
the value function corresponding to the “all-default” configuration is independent of all others, while the
value function corresponding to the “all-alive” configuration depends on the value functions associated to
all possible default states in the lattice. We rigorously show that each value function corresponds to the
unique positive solution to a HJB equation, whose solution can be fully characterized in terms of solutions
to inhomogeneous Bernoulli’s type ODEs. Despite the interacting property prevents obtaining fully explicit
expressions, we can still uniquely characterize the value function and the corresponding optimal CDS strategy
via an implicit relation involving only solutions to ODE’s and non-linear equations.

We provide an economic analysis to assess the impact of default contagion on the optimal CDS strategy.
Considering a portfolio consisting of a stock index and two credit default swaps, we illustrate how contagion
effects impact optimal portfolio allocations. We find that the default of a very risky name significantly alters
the amount of wealth allocated to the CDS referencing the surviving name, and may even induce a change in
the directionality of the strategy (from long credit to short credit). Moreover, if default risk premium,
risk neutral and historical default intensities are time invariant, the investor trades more aggressively if
the planning horizon is higher. Further, we find that default contagion effects may dominate on the current
default state: under the situation when one name is very risky, the allocation strategy on the CDS referencing the safest name becomes highly dependent on the post-default intensity of the safest name (i.e. the one after default of the risky name) and only mildly dependent on its current pre-default intensity (i.e. the one before default of the risky name). This indicates the important role played by default contagion in determining the optimal investor strategies.

The rest of the paper is organized as follows. Section 2 develops the market and default model. Section 3 formulates the utility maximization problem. Section 4 analyzes the optimal investment strategy and proves existence and uniqueness of a positive solution to the resulting HJB equation. Section 5 presents a rigorous proof that the solutions of the HJB equations associated with the different default configurations correspond with the value functions. Section 6 develops a numerical analysis to assess the impact of default contagion on the optimal strategies. Additional technical proofs are delegated to an Appendix.

2 The Market Model

We consider $M \geq 2$ reference entities in the underlying CDS portfolio. The default state is described by a $M$-dimensional default indicator process $H(t) = (H_1(t), \ldots, H_M(t))$ with $t \geq 0$, supported by a filtered probability space $(\Omega, \mathcal{G}, \mathbb{P})$. Here, $\mathbb{P}$ denotes the physical probability measure and $\mathbb{E}^\mathbb{P}$ the expectation operator w.r.t. $\mathbb{P}$. The state space of the default indicator process $H = (H(t); \ t \geq 0)$ is given by $\mathcal{S} = \{0,1\}^M$, where $H_i(t) = 1$ if the name $i$ has defaulted by time $t$ and $H_i(t) = 0$ otherwise. The default time of the $i$-th name in the portfolio is given by

$$\tau_i = \inf\{t \geq 0; \ H_i(t) = 1\} \quad i = 1, \ldots, M.$$ 

Hence, we have $H_i(t) = \mathbb{1}_{\{\tau_i \leq t\}}$, where $t \geq 0$.

We model default contagion through a Markovian model with interacting intensities. The default indicator process $H$ is assumed to follow a continuous-time Markov chain on $\mathcal{S} = \{0,1\}^M$, where $H(t)$ transits to neighbouring states $H'(t) := (H_1(t), \ldots, H_{i-1}(t), 1 - H_i(t), H_{i+1}(t), \ldots, H_M(t))$ at rate $1_{\{H_i(t) = 0\}} h_i^H(H(t))$. Here $h_i^H(z)$ ($i = 1, \ldots, M$) are arbitrary measurable functions defined on $z \in \mathcal{S}$. We refer the reader to Frey and Backhaus (2004) for explicit probabilistic models under this setup. Hence, the $M$-dimensional default indicator process $H$ admits the following $\mathbb{P}$-infinitesimal generator given by

$$\mathcal{A}^\mathbb{P} g(z) = \sum_{j=1}^{M} (1 - z_j) h_j^H(z) \left[ g(z') - g(z) \right], \quad z = (z_1, \ldots, z_M) \in \mathcal{S},$$

where $g(z)$ is an arbitrary measurable function defined on $z \in \mathcal{S}$, and the vector

$$z^j := (z_1, \ldots, z_{j-1}, 1 - z_j, z_{j+1}, \ldots, z_M), \quad j = 1, \ldots, M.$$ 

Let the filtration $\mathcal{G}^\mathbb{P}_t = \sigma(H(s); \ s \leq t) \subset \mathcal{G}$, for each $t > 0$. Using the Dynkin’s formula (see (10.13) in Rogers and Williams (2000), pag. 254), and choosing $g(z) = z_i$, where $i \in \{1, \ldots, M\}$, we have that

$$\mathbb{E}^\mathbb{P}_t (t) := H_i(t) - \int_0^t (1 - H_i(s)) h_i^H(H(s)) \, ds, \quad t \geq 0$$

is a $(\mathbb{P}, \mathcal{G}^\mathbb{P}_t)$-martingale.

We consider a frictionless financial market consisting of a risk-free money market account, a stock index, and $M$ credit default swaps. We have that $B_t = e^{rt}$ is the money market account at $t$, with $r > 0$ being the constant interest rate. The $\mathbb{P}$ dynamics of the stock index at time $t$ is given by a geometric Brownian motion, namely

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t,$$

where $W = (W_t; \ t \geq 0)$ is a $\mathbb{P}$-Brownian motion. We let $\mathcal{G}^W_t = \sigma(W_s; \ 0 \leq s \leq t)$. The market filtration is then given by $\mathcal{G} = (\mathcal{G}_t; \ t \geq 0)$, where $\mathcal{G}_t = \mathcal{G}^\mathbb{P}_t \lor \mathcal{G}^W_t$, after completion and regularization on the right, see Belanger et al. (2004).

Unlike the previous two securities, whose dynamics have been written under the historical measure, the CDS prices are defined under a suitably chosen risk-neutral measure $Q$. In the sequel of the paper, to lighten notation we will omit the dependence on $Q$, when denoting quantities under the risk-neutral measure, for instance $\mathbb{E}$ will be intended under $Q$. Next, we provide a formula which allows identifying the risk-neutral probability measure $Q$ from the historical measure $\mathbb{P}$. Let $\lambda_i(z)$ be an arbitrary bounded
measurable function defined on $\mathbf{z} \in \mathbf{S}$, which takes values on $(-1, \infty)$, where $i \in \{1, \ldots, M\}$. Assume that the process $X = (X_t; t \geq 0)$ satisfies the following SDE given by

$$\frac{dX_t}{X_t} = \sum_{i=1}^{M} \lambda_i(H(t^-))d\xi_i^P(t), \quad X_0 = 1, \quad (5)$$

where the $(\mathbb{P}, \mathcal{G}_t^H)$-default martingale process $\xi_i^P = (\xi_i^P(t); \ t \geq 0)$ is defined by (3). Then we have

**Lemma 2.1.** For $T > 0$, define a new probability measure $\mathbb{Q} \ll \mathbb{P}$ on $\mathcal{G}_T$ by

$$d\mathbb{Q} = X_Td\mathbb{P}.$$  

Then, for each $i = 1, \ldots, M$,

$$\xi_i(t) := H_i(t) - \int_0^t (1 - H_i(s))h_i(H(s))ds, \quad t \geq 0 \quad (6)$$

is a $(\mathbb{Q}, \mathcal{G}_t^H)$-martingale, where the $\mathbb{Q}$-default intensity of the $i$-th default indicator $H_i(t)$ is given by

$$h_i(\mathbf{z}) := h_i^P(\mathbf{z})(1 + \lambda_i(\mathbf{z})), \quad \mathbf{z} \in \mathbf{S}.$$  

The proof of Lemma 2.1 is reported in the Appendix. We consider $M$ stylized credit default swaps, parameterized by $(\{L_i, \nu_i\})_{i=1}^{M}$, where $L_i, \nu_i$ are positive constants. Here, $\nu_i$ is the continuously paid spread premium associated with the $i$-th reference entity, while $L_i$ is the loss paid by the protection seller when the $i$-th name defaults. For each CDS, the payoffs are seen from the point of view of the protection buyer. The market price of the $i$-th CDS is given by

$$CDS_i(t) = (1 - H_i(t))\hat{\Phi}^L_{i}(t, H(t))), \quad 0 \leq t \leq T_i, \quad (7)$$

where $\hat{\Phi}^L_{i}(t, H(t))$ denotes the pre-default price of the $i$-th CDS contract, given by

$$\hat{\Phi}^L_{i}(t, H(t)) = \mathbb{E} \left[ L_i \int_t^{T_i} e^{-\int_s^t \nu_i rds}dH_i(u) - \nu_i \int_t^{T_i} e^{-\int_s^t \nu_i rds}(1 - H_i(u))du \bigg| \mathcal{G}_t^H \right].$$

The Markovianity property of the default framework allows obtaining explicit expressions for the pre-default price of each CDS contract. Concretely, for $i = 1, \ldots, M$, define the functions

$$\Phi_{i}^{(1)}(t, \mathbf{z}) := \mathbb{E} \left[ \int_t^{T_i} e^{-\int_s^t \nu_i rds}(1 - H_i(u))du \bigg| H(t) = \mathbf{z} \right],$$
$$\Phi_{i}^{(2)}(t, \mathbf{z}) := \mathbb{E} \left[ H_i(T_i)e^{-\int_t^{T_i} r(1 - H_i(u))du} \bigg| H(t) = \mathbf{z} \right]. \quad (9)$$

and recall the definition of $\mathbf{z}^i$ given in Eq. (2). We then have the following result, whose proof is in the Appendix.

**Lemma 2.2.** For $\mathbf{z} \in \{0, 1\}^M$, let

$$\hat{\Phi}_{i}^{L}(t, H(t)) := L_i \Phi_{i}^{(2)}(t, \mathbf{z}) - \nu_i \Phi_{i}^{(1)}(t, \mathbf{z}), \quad i = 1, \ldots, M.$$  

It holds that, for each $i = 1, \ldots, M$,

$$\hat{\Phi}_{i}^{L}(t, H(t)) = \hat{\Phi}_{i}^{L}(0, H(0)) + \int_0^t \left( r(1 - H_i(s))\hat{\Phi}_{i}^{L}(s, H(s)) - r\nu_i H_i(s)\Phi_{i}^{(1)}(s, H(s)) + \nu_i(1 - H_i(s)) \right)ds + \sum_{j=1}^{M} \int_0^t \left[ \hat{\Phi}_{i}^{L}(s, H^j(s)) - \hat{\Phi}_{i}^{L}(s, H(s)) \right]d\xi_j(s), \quad (11)$$

where $\xi_j(t) := H_j(t) - \int_0^t (1 - H_j(s))h_j(H(s))ds$ is a $\mathbb{Q}$-martingale for each $j = 1, \ldots, M$.  

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 Explicit expressions for $\hat{\phi}^L_{i,t}(t,z)$, obtained by solving the associated Feynman-Kac equations, are provided in Lemma A.1 and Corollary A.2 in the Appendix.

The $\mathbb{Q}$-dynamics followed by the price process of the $i$-th CDS is given in the following proposition, whose proof is reported in the Appendix.

**Proposition 2.3.** The $\mathbb{Q}$-dynamics of the $i$-th CDS price is given by

$$
dCS_t^{(i)} = (1 - H_i(t)) \left[ r \hat{\phi}^L_{i,t}(t,H(t)) + (v_t - h_i(H(t))L) \right] dt - \hat{\phi}^L_{i,t}(t,H(t)) d\xi_i(t) + (1 - H_i(t)) \sum_{j \neq i} \left[ \hat{\phi}^L_{i,t}(t,H^j(t)) - \hat{\phi}^L_{i,t}(t,H(t)) \right] dt \nonumber $$

$$+ (1 - H_i(t)) \sum_{j \neq i} \left[ \hat{\phi}^L_{i,t}(t,H^j(t)) - \hat{\phi}^L_{i,t}(t,H(t)) \right] d\xi_j(t),
$$

$$
CDS_0^{(i)} = (1 - H_i(0)) \hat{\phi}^L_{i,0}(0, H(0)) \in [-\nu_i, L_i].
$$

Recall that $h^P_i(z)$ is the $\mathbb{P}$-default intensity with $z \in S$ as in Lemma 2.1, i.e. the following process $\xi^P_i(t)$ is a $\mathbb{P}$-martingale:

$$
\xi^P_i(t) := H_i(t) - \int_0^t (1 - H_i(s)) h^P_i(H(s)) ds, \quad t \geq 0.
$$

Then, we have the following

**Corollary 2.4.** The $\mathbb{P}$-dynamics of the $i$-th CDS price is given by

$$
dCS_t^{(i)} = (1 - H_i(t)) \left[ r \hat{\phi}^L_{i,t}(t,H(t)) + (v_t - h_i(H(t))L) \right] dt + \hat{\phi}^L_{i,t}(t,H(t)) \left( h_i(H(t)) - h^P_i(H(t)) \right) dt \nonumber $$

$$+ (1 - H_i(t)) \sum_{j \neq i} \left[ \hat{\phi}^L_{i,t}(t,H^j(t)) - \hat{\phi}^L_{i,t}(t,H(t)) \right] \left( h_j(H(t)) - h^P_j(H(t)) \right) dt \nonumber $$

$$- \hat{\phi}^L_{i,t}(t,H(t)) d\xi_i(t) + (1 - H_i(t)) \sum_{j \neq i} \left[ \hat{\phi}^L_{i,t}(t,H^j(t)) - \hat{\phi}^L_{i,t}(t,H(t)) \right] d\xi_j(t) \nonumber $$

$$
CDS_0^{(i)} = (1 - H_i(0)) \hat{\phi}^L_{i,0}(0, H(0)) \in [-\nu_i, L_i].
$$

Using the above $\mathbb{P}$ dynamics, it follows that the jump of the $i$-th CDS price is given as follows. If the $i$-th name defaults at $t$, $\Delta CDS_t^{(i)} = -\hat{\phi}^L_{i,t}(t,H(t))$. If the $\ell \neq i$-name defaults at $t$, then $\Delta CDS_t^{(i)} = \hat{\phi}^L_{i,t}(t,H^\ell(t)) - \hat{\phi}^L_{i,t}(t,H(t))$.

**Remark 2.5.** In the case of a single CDS, the default contagion term

$$(1 - H_i(t)) \sum_{j \neq i} \left[ \hat{\phi}^L_{i,t}(t,H^j(t)) - \hat{\phi}^L_{i,t}(t,H(t)) \right] d\xi_j(t)$$

in (12) disappears from the dynamics. In such a case, the dynamics of the CDS reduces to the one in Bielecki et al. (2008b) (see Eq. (32) in pag. 2506).

### 3 The Utility Maximization Problem

We consider an investor who wants to maximize her power utility from terminal wealth at time $T$ by dynamically allocating her wealth into the money market account, the stock index, and $M$ credit default swaps. The investor does not have intermediate consumption nor capital income to support her purchase of financial assets. For each $i = 1, \ldots, M$, denote by $\phi^{CDS}_i(t)$ the number of shares of the $i$-th CDS that the investor buys ($\phi^{CDS}_i(t) > 0$) or sells ($\phi^{CDS}_i(t) < 0$) at time $t$. Similarly, $\phi^S(t)$ and $\phi^B(t)$ denote, respectively, the number of shares invested in the stock index and in money market account at $t$. The process $\phi = (\phi^{CDS}_1(t), \phi^S(t), \phi^B(t); t \geq 0)$ with $\phi^{CDS}_i(t) = (\phi^{CDS}_i(t), \ldots, \phi^{CDS}_M(t))$ is called a portfolio process. The wealth process associated to the portfolio process $\phi = (\phi^{CDS}_i(t), \phi^S(t), \phi^B(t); t \geq 0)$, denoted by $V(t)$, is given by

$$V_i(\phi) = \sum_{i=1}^{M} \phi^{CDS}_i(t) CDS_t^{(i)} + \phi^S(t) S_t + \phi^B(t) B_t, \quad t \geq 0,
$$

$$
V_i(\phi) = \sum_{i=1}^{M} \phi^{CDS}_i(t) CDS_t^{(i)} + \phi^S(t) S_t + \phi^B(t) B_t, \quad t \geq 0
$$

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As usual, we require the portfolio process $\phi$ to be $\mathcal{G}$-adapted. Following Bielecki et al. (2008b), see also Bielecki et al. (2008a), a $\mathcal{G}$-adapted portfolio process $\phi = (\phi^{CDS}(t), \phi^{d}(t), \phi^{B}(t); t \geq 0)$ is said to be self-financing if $V_t(\phi) = V_0(\phi) + G_t(\phi)$, where the gains process $G(\phi)$ is given by

$$G_t(\phi) = \sum_{i=1}^{M} \int_{0}^{t} \phi^{CDS}_i(s) d(CDS^{(i)}_s) + D^{(i)}_s + \int_{0}^{t} \phi^{S}(s) dS_s + \int_{0}^{t} \phi^{B}(s) dB_s, \quad t \geq 0,$$

with $D^{(i)}_i = (D^{(i)}_t; t \geq 0)$ being the dividend process corresponding to the $i$-th CDS. The latter is a finite variation process given by

$$D^{(i)}_t = L_i H_i(t) - \nu_i \int_{0}^{t} (1 - H_i(s)) ds, \quad t \geq 0,$$

where $D^{(i)}_0 = 0$, where $i = 1, \ldots, M$. Moreover, for $0 \leq t \leq T$, we define $\pi^{S}_i := \frac{\phi^{S}_i}{V_t(\phi)}$ and $\pi^{B}_i := \frac{\phi^{B}_i}{V_t(\phi)}$ to denote, respectively, the proportions of wealth invested in stock and in the money market account. For the CDS investment strategy, it turns out more convenient to develop the analysis using the number of shares. Next, we provide a definition.

**Definition 3.1.** The admissible control set $\mathcal{U}_t(v, z)$, $t \geq 0$, is a suitable class of $\mathcal{G}$-predictable locally bounded feedback trading strategies given by

$$\phi(u) := (\phi^{CDS}(u), \pi^{S}_u) := (\phi^{CDS}(u), \pi^{S}_u(V^t_u(\phi), Z^t_u(\phi), \pi^{S}_u(V^t_u(\phi), Z^t_u(\phi))), \quad u \in [t, T],$$

where $\phi^{CDS}(u) = (\phi^{CDS}_1(u), \ldots, \phi^{CDS}_M(u))$, and $\phi^{CDS}_i(u)$ denotes the pre-default number of shares of the $i$-th CDS contract

$$\phi^{CDS}_i(u) = (1 - H_i(u)) \phi^{CDS}_i(u) = (1 - H_i(u)) \phi^{CDS}_i(u, V^t_u(\phi)).$$

Here $(V^t_u(\phi); u \in [t, T])$ denotes the positive wealth process associated with the strategy $\phi$ when $V_t = v$ and $H(t) = z$. Throughout the paper, a trading strategy satisfying the above conditions is said to be $t$-admissible w.r.t. the initial conditions $V_t = v$ and $H(t) = z$.

We remark our slight abuse of notation in first using $\phi$ to denote number of shares in both stocks and CDSs, and then using it to denote number of shares in CDSs and proportion of wealth in the stock index. We do so to avoid introducing too much notation.

Let $\phi \in \mathcal{U}_0 = \mathcal{U}_0(v, z)$. Then we can describe the wealth process as, $V_0(\phi) = v$, and

$$dV_t(\phi) = \sum_{i=1}^{M} \phi^{CDS}_i(t-) dCDS^{(i)}_t + \sum_{i=1}^{M} \phi^{CDS}_i(t-) dD^{(i)}_t + V_{-}(\phi) \left( \pi^{S}_t \frac{dS_t}{S_t} + \pi^{B}_t \frac{dB_t}{B_t} \right)$$

$$= \sum_{i=1}^{M} \phi^{CDS}_i(t) \left\{ rCDS^{(i)}_t + (CDS^{(i)}_t - L_i) h_i(H(t)) \right\} dt$$

$$- \sum_{j \neq i} \left[ \Phi^{L,v}_i(t, H^j(t)) - CDS^{(i)}_t \right] (1 - H_j(t)) h_j(H(t)) \right\} dt$$

$$+ \sum_{i=1}^{M} \phi^{CDS}_i(t-) \left( L_i - CDS^{(i)}_t \right) dH_i(t)$$

$$+ \sum_{i=1}^{M} \phi^{CDS}_i(t) \sum_{j \neq i} \left[ \Phi^{L,v}_i(t, H^j(t)) - CDS^{(i)}_t \right] dH_j(t)$$

$$+ V_0(\phi) \left[ \mu \pi^{S}_t dt + \sigma \pi^{S}_t dW_t + r \pi^{B}_t dt \right]$$

$$= \left\{ V_0(\phi) \left[ r + (\mu - r) \pi^{S}_t \right] + \sum_{i=1}^{M} \phi^{CDS}_i(t) \left[ (CDS^{(i)}_t - L_i) h_i(H(t)) \right]$$

$$- \sum_{j \neq i} \left( \Phi^{L,v}_i(t, H^j(t)) - CDS^{(i)}_t \right) (1 - H_j(t)) h_j(H(t)) \right\} dt$$

$$+ \sum_{i=1}^{M} \phi^{CDS}_i(t-) \left( L_i - CDS^{(i)}_t \right) dH_i(t)$$

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\[ + \sum_{i=1}^{M} \phi_i^{CDS}(t-) \sum_{j\neq i} \left( \tilde{\Phi}_i^{L,\nu}(t, H^j(t-)) - CDS_i(t-) \right) dH_j(t) + \sigma V_i(\phi) \pi_i^S dW_t, \]  

where we used the equalities \( CDS_i(t-) = (1 - H_i(t)) \tilde{\Phi}_i^{L,\nu}(t, H(t)) \) (see (7)), \( \phi_i^{CDS}(t) = \phi_i^{CDS}(t)(1 - H_i(t)) \), and the self-financing condition

\[ \sum_{i=1}^{M} \phi_i^{CDS}(t) CDS_i + V_i(\phi) \pi_i^B = V_i(\phi) \left( 1 - \pi_i^S \right). \]

Then, we can conclude that

**Lemma 3.1.** For any admissible strategy \( \phi = (\phi^{CDS}, \pi^S) \in \mathcal{U}_0 \), the \( \mathbb{P} \)-dynamics of the wealth process \( V = (V_t(\phi); \ t \geq 0) \) is given by

\[
dV_t(\phi) = \left\{ V_t(\phi) \left[ r + (\mu - r) \pi_i^S \right] + \sum_{i=1}^{M} \phi_i^{CDS}(t) \left[ \left( \tilde{\Phi}_i^{L,\nu}(t, H(t)) - L_i \right) h_i(H(t)) \right. \right.
\]
\[ - \sum_{j \neq i} \left( \tilde{\Phi}_i^{L,\nu}(t, H^j(t)) - \tilde{\Phi}_i^{L,\nu}(t, H(t)) \right) (1 - H_j(t)) h_j(H(t)) \left. \right\} dt \]
\[ + \sum_{i=1}^{M} \phi_i^{CDS}(t-) \left( L_i - \tilde{\Phi}_i^{L,\nu}(t, H(t)) \right) dH_i(t) \]
\[ + \sum_{i=1}^{M} \phi_i^{CDS}(t-) \sum_{j \neq i} \left( \tilde{\Phi}_i^{L,\nu}(t, H^j(t)) - \tilde{\Phi}_i^{L,\nu}(t, H(t)) \right) dH_j(t) + \sigma V_t(\phi) \pi_i^S dW_t, \]

where \( W = (W_t; \ t \geq 0) \) is the \( \mathbb{P} \)-Brownian motion given by (4).

**Remark 3.2.** From the wealth dynamics given by (18), it can be seen that the jump \( \Delta V_u = V_u - V_{u-} \) at time \( u \) is given by

\[ \Delta V_u = \sum_{i=1}^{M} \phi_i^{CDS}(u, V_{u-}, H(u-)) \left( L_i - \tilde{\Phi}_i^{L,\nu}(u, H(u-)) \right) \Delta H_i(u) \]
\[ + \sum_{i=1}^{M} \phi_i^{CDS}(u, V_{u-}, H(u-)) \sum_{j \neq i} \left( \tilde{\Phi}_i^{L,\nu}(u, H^j(u-)) - \tilde{\Phi}_i^{L,\nu}(u, H(u-)) \right) \Delta H_j(u). \]

For \( (V_u; \ u \in [t, T]) \) to be strictly positive, it is necessary and sufficient that \( \Delta V_u > -V_{u-} \) a.s. for any \( u \in [t, T] \), see (cf. (Jacod and Shiryaev, 2003, Theorem 4.61)). We will account for this condition in Section 4.1, when we characterize the optimal CDS investment strategy under the different default configurations.

For \( \phi = (\phi^{CDS}, \pi^S) \), define the operators, depending on \( \phi \), acting on the smooth function \( w(t, v, z) \) as

\[
\mathcal{L}_w(t, v, z) = \left\{ \sum_{i=1}^{M} \phi_i^{CDS}(1 - z_i) \left[ \left( \tilde{\Phi}_i^{L,\nu}(t, z) - L_i \right) h_i(z) \right. \right. \\
\left. - \sum_{j \neq i} \left( \tilde{\Phi}_i^{L,\nu}(t, z^j) - \tilde{\Phi}_i^{L,\nu}(t, z) \right) (1 - z_j) h_j(z) \right] + v [r + (\mu - r) \pi^S] \right\} w(t, v, z) \\
+ \frac{1}{2} \sigma^2 (\pi^S)^2 v^2 w_{vv}(t, v, z), \\
\mathcal{L}_z w(t, v, z) = \sum_{i=1}^{M} \left[ w(t, v + \phi_i^{CDS}(1 - z_i)(L_i - \tilde{\Phi}_i^{L,\nu}(t, z)), z) - w(t, v, z) \right] (1 - z_i) h_i^z(z) \\
+ \sum_{i=1}^{M} \left\{ \sum_{j \neq i} \left[ w(t, v + \phi_i^{CDS}(1 - z_i)(\tilde{\Phi}_i^{L,\nu}(t, z^j) - \tilde{\Phi}_i^{L,\nu}(t, z)), z^j) - w(t, v, z) \right] (1 - z_j) h_j^z(z) \right\}. 
\]

**Remark 3.3.** Let \( \varphi^R(t, v, z) \) be a smooth function, where \( (t, v, z) \in [0, T] \times \mathbb{R}_+ \times \{0, 1\}^M \). Then using Ito’s formula, it follows that

\[
\varphi^R(t, V_t, H(t)) = \varphi^R(t, v, z) + \int_0^t \varphi^R_s(V_s, H(s))ds + \int_0^t \varphi^R_s(s, V_s, H(s))dV^c_s. 
\]
positive solution to the HJB equation (23) in Section 4.2. We then prove existence and uniqueness of a solution to the Hamiltonian-Jacobi-Bellman equation exists. We analyze the optimal investment strategy in credit default swaps in Section 4.1, assuming that a smooth utility if \( \gamma \in (0, 1) \) is the risk-aversion parameter.

Our goal is to maximize the objective functional \( J^T(\phi; v, z) := \mathbb{E}^P \left[ U(V_T(\phi)) \right] | V_0 = v, H(0) = z \),

where \( V_t = V_t(\phi) \) is the wealth process given by (18) and \( V^C \) denotes the continuous part of the wealth process.

Next, we formulate the portfolio optimization problem. To this purpose, we define the objective functional

\[
J^T(\phi; v, z) := \mathbb{E}^P \left[ U(V_T(\phi)) \right] | V_0 = v, H(0) = z,
\]

where \( T \in (0, \min(T_1, \ldots, T_M)) \) is the time horizon. The utility function \( U : [0, \infty) \rightarrow [0, \infty) \) is chosen to be the power utility \( U(v) = \frac{v^\gamma}{\gamma} \), where \( \gamma \in (0, 1) \) is the risk-aversion parameter.

Our goal is to maximize the objective functional \( J^T(\phi; v, z) \) across the class of admissible strategies \( \phi \in \mathcal{U}_t \) defined in 3.1. Hence, we consider the following dynamical optimization problem:

\[
w^T(t, v, z) := \sup_{\phi \in \mathcal{U}_t = \mathcal{U}_t(v, z)} \mathbb{E}^P \left[ U(V_T(\phi)) \right] | V_t = v, H(t) = z,
\]

where \( V_t(\phi) \) is given by Lemma 3.1.

To start with, let us assume that \( w^T(t, v, z) \) is \( C^1 \) in \( t \), and \( C^2 \) in \( v \) for each \( z \in \mathcal{S} \). Then using Itô’s formula along the lines of Remark 3.3, we have that, for any \( u > t \geq 0 \),

\[
w^T(u, v, \phi, H(u)) = w^T(t, V_t(\phi), H(t)) + \int_t^u \left( \frac{\partial}{\partial t} + L \right) w^T(s, V_s(\phi), H(s)) \, ds + \mathcal{M}(u) - \mathcal{M}(t),
\]

where the operator \( L = L_u + L_z \), and the process \( \mathcal{M} = (\mathcal{M}(t); t \geq 0) \) is a \( \mathbb{P} \)-local martingale. Next, for \( 0 \leq t < u \leq T \), by virtue of the dynamics programming principle, we expect that

\[
w^T(t, V_t(\phi), H(t)) = \sup_{\phi \in \mathcal{U}_t} \mathbb{E}^P_t \left[ w^T(u, V_u(\phi), H(u)) \right],
\]

where \( \mathbb{E}^P_t[\cdot] := \mathbb{E}^P[|\mathcal{G}_t|] \).

Therefore, we obtain \( \mathbb{E}^P \left[ \int_t^u \left( \frac{\partial}{\partial t} + L \right) w^T(s, V_s(\phi), H(s)) \, ds \right] \leq 0 \), with the inequality becoming an equality if \( \phi = \phi^* \), where \( \phi^* \) denotes the optimum. This leads to the following HJB equation:

\[
\sup_{\phi \in \mathcal{U}_t} \left( \frac{\partial}{\partial t} + L \right) w^T(t, v, z) = 0, \quad w^T(T, v, z) = U(v).
\]

### 4 Optimal Investment Strategies and HJB Equations

We analyze the optimal investment strategy in credit default swaps in Section 4.1, assuming that a smooth solution to the Hamiltonian-Jacobi-Bellman equation exists. We then prove existence and uniqueness of a positive solution to the HJB equation (23) in Section 4.2.
4.1 Optimal Strategies

We start analyzing the optimal strategy, denoted by \( \phi^* \), using the first-order condition. For future purposes, it is convenient to introduce \( \phi^*_i^{CDS} \), defined by \( \phi^*_i^{CDS} = v \phi_i^{CDS} \), for \( i = 1, \ldots, M \). The optimum can be written as \( \phi^* = (\phi^{CDS,*}_1, \pi^{S,*}) \), where \( \phi^{CDS,*}_i = (\phi_i^{CDS,*}, \ldots, \phi_M^{CDS,*}) \). Define
\[
 f^T(\phi; t, v, z) := (L_e + L_J) w^T(\phi; t, v, z),
\]
For each \( i = 1, \ldots, M \), the first-order condition given by
\[
 f^{T,i}_i(\phi; t, v, z) := \frac{\partial f^T(\phi; t, v, z)}{\partial \phi_i^{CDS}} = 0,
\]
yields the optimum \( \phi^{CDS,*}_i \), i.e. \( f^{T,i}_i(\phi^*; t, v, z) = 0 \). It can be seen that
\[
 f^{T,i}_i(\phi^*; t, v, z) = vw^T(t, v, z)(1 - z) \left[ \left( \hat{\phi}_i^{L,\nu}(t, v) - L_i \right) h_i(z) - \sum_{j \neq i} \left( \hat{\phi}_i^{L,\nu}(t, v^j) - \hat{\phi}_i^{L,\nu}(t, z^j) \right) (1 - z_j)h_j(z) \right] 
\]
\[
+ vw^T \left( t, v + v \phi_i^{CDS,*}(1 - z_i)(L_i - \hat{\phi}_i^{L,\nu}(t, z)), z^i \right) (L_i - \hat{\phi}_i^{L,\nu}(t, z))(1 - z_i)h_i^P(z) 
\]
\[
+ \sum_{j \neq i} vw^T \left( t, v + v \phi_i^{CDS,*}(1 - z_i) \left( \hat{\phi}_i^{L,\nu}(t, z^j) - \hat{\phi}_i^{L,\nu}(t, z^j) \right), z^j \right) (1 - z_j)h_j^P(z)(1 - z_i) 
\]
\[
\times \left( \hat{\phi}_i^{L,\nu}(t, z) - \hat{\phi}_i^{L,\nu}(t, z) \right). 
\]
(25)

Using the first-order condition (24), we can write that the optimum \( \phi_i^{CDS,*} \) as
\[
 \phi_i^{CDS,*} = \phi_i^{CDS,*}(t, v, z).
\]
We postulate (and later show), that the value function \( w^T \) is separable, and given by
\[
 w^T(t, v, z) = v^T B(t, z), \quad (26)
\]
where \( B(t, z) \) satisfies an appropriate ODE, analyzed in the next section. For brevity, we introduce the notation \( B(t, z) := B(t, z) \). We now discuss the existence and uniqueness of the optimal strategy \( \phi^{CDS,*}(t, z) \). We separate the analysis into three cases, namely (1) all names alive, (2) some names defaulted, (3) all names defaulted. We have

1. All names are alive, i.e. \( z = 0 \). From (25), for \( i = 1, \ldots, M \), the optimum \( \phi_i^{CDS}(t, 0) \) satisfies
   \[
   0 = vw^T(t, v, 0) \left[ \left( \hat{\phi}_i^{L,\nu}(t, 0) - L_i \right) h_i(0) - \sum_{j \neq i} \left( \hat{\phi}_i^{L,\nu}(t, 0^j) - \hat{\phi}_i^{L,\nu}(t, 0^j) \right) h_j(0) \right] 
   \]
   \[
   + vw^T \left( t, v + v \phi_i^{CDS,*}(0)(L_i - \hat{\phi}_i^{L,\nu}(t, 0^i)), 0^i \right) (L_i - \hat{\phi}_i^{L,\nu}(t, 0^i))h_i^P(0) 
   \]
   \[
   + \sum_{j \neq i} vw^T \left( t, v + v \phi_i^{CDS,*}(0)(0^j) \left( \hat{\phi}_i^{L,\nu}(t, 0^j) - \hat{\phi}_i^{L,\nu}(t, 0^j) \right), 0^j \right) h_j^P(0) \left( \hat{\phi}_i^{L,\nu}(t, 0^j) - \hat{\phi}_i^{L,\nu}(t, 0^j) \right). 
   \]

Using the representation of \( w^T(t, v, 0) = v^T B^0(0) \), the optimum \( \phi_i^{CDS,*}(t, 0) \) satisfies
\[
 0 = V_i(t) B^{0}(0) + B^{0}(0)(L_i - \hat{\phi}_i^{L,\nu}(t, 0^i))h_i^P(0) \left[ 1 + \phi_i^{CDS,*}(t, 0)(L_i - \hat{\phi}_i^{L,\nu}(t, 0^i)) \right]^{-1} 
\]
\[
+ \sum_{j \neq i} B^{0}(0) \left( \hat{\phi}_i^{L,\nu}(t, 0^j) - \hat{\phi}_i^{L,\nu}(t, 0^j) \right) h_j^P(0) \left[ 1 + \phi_i^{CDS,*}(t, 0) \left( \hat{\phi}_i^{L,\nu}(t, 0^j) - \hat{\phi}_i^{L,\nu}(t, 0^j) \right) \right]^{-1} 
\]
where
\[
 V_i(t) := \left( \hat{\phi}_i^{L,\nu}(t, 0) - L_i \right) h_i(0) - \sum_{j \neq i} \left( \hat{\phi}_i^{L,\nu}(t, 0^i) - \hat{\phi}_i^{L,\nu}(t, 0^i) \right) h_j(0). 
\]

Define the \( M + 1 \) dimensional vector
\[
 B^{0}(0) := \left[ B^{0}(0), B^{0}(0), B^{0}(0) ; j \in \{1, \ldots, M\}, j \neq i \right]^T. 
\]
Moreover, for $i = 1, \ldots, M$, define
\[
g_i\left(\phi_i^{CDS}, t, B^{(i)}\right)
= V_i(t)B_1^{(i)}(L_i - \Phi_i^{L,t}(t,0))h_i^0(t,0)\left[1 + \phi_i^{CDS}(L_i - \Phi_i^{L,t}(t,0))\right]^{-1}
+ \sum_{j \neq i} B_j^{(i)}\left(\Phi_i^{L,t}(t,0^j) - \Phi_i^{L,t}(t,0)\right)h_j^0(t,0)\left[1 + \phi_i^{CDS}(\Phi_i^{L,t}(t,0^j) - \Phi_i^{L,t}(t,0))\right]^{-1}.
\]

Our goal is to find the optimum $\phi_i^{CDS,*}(t,0)$ satisfying the following conditions, for each $i = 1, \ldots, M$,
\[
\phi_i^{CDS,*}(t,0) > -\frac{1}{L_i - \Phi_i^{L,t}(t,0)} =: M_i^{(1)}(t), \text{ and}
\phi_i^{CDS,*}(t,0)\left(\Phi_i^{L,t}(t,0^i) - \Phi_i^{L,t}(t,0)\right) > -1, \forall j \neq i.
\]

Without loss of generality, assume that there exists $j_1, \ldots, j_m$ for some $m = 1, \ldots, M$ such that $\Phi_i^{L,t}(t,0^{j_k}) < \Phi_i^{L,t}(t,0)$ for $k = 1, \ldots, m$, where $j_1 \neq \cdots \neq j_m$ belonging to $\{1, \ldots, M\} \setminus \{i\}$. Then, for $k = m + 1, \ldots, M$, it holds that $\Phi_i^{L,t}(t,0^{j_k}) > \Phi_i^{L,t}(t,0)$.

Note that
\[
\max\left\{ M_i^{(1)}(t), \frac{1}{\Phi_i^{L,t}(t,0) - \Phi_i^{L,t}(t,0^j)}; \ k = m + 1, \ldots, M \right\} = M_i^{(1)}(t),
\]

since $\Phi_i^{L,t}(t,0) < L_i$ for all $j$ using Lemma A.1. Define
\[
M_i^{(2)}(t) := \min\left\{ \frac{1}{\Phi_i^{L,t}(t,0) - \Phi_i^{L,t}(t,0^j)}; \ k = 1, \ldots, m \right\},
\]

where $M_i^{(2)}(t) = +\infty$ if the set $\{j\}$ of the r.h.s. of the above equality is empty (i.e., $m = 0$). Hence, for each $i = 1, \ldots, M$, the optimum $\phi_i^{CDS}(t,0)$ should satisfy
\[
M_i^{(1)}(t) < \phi_i^{CDS}(t,0) < M_i^{(2)}(t).
\]

The following lemma, whose proof is reported in the Appendix A, establishes existence and uniqueness of a unique solution $\phi_i^{CDS}$ satisfying the first-order condition (27), under the constraints specified by (30).

**Lemma 4.1.** For each $i = 1, \ldots, M$, there exists a unique optimum $\phi_i^{CDS,*}(t,0)$ satisfying the equation
\[
g_i\left(\phi_i^{CDS}, t, B^{(i)}\right) = 0,
\]

for each $(t, B^{(i)}) \in [0,T] \times \mathbb{R}_+^{M+1}$, subject to the condition (30). Moreover, the optimum $\phi_i^{CDS,*}(t,0)$, viewed as a function of $(t, B^{(i)}) \in [0,T] \times \mathbb{R}_+^{M+1}$, is continuous in $(t, B^{(i)})$ for each $i = 1, \ldots, M$.

2. Some names are defaulted, i.e. $z = (0^{j_1})^{\cdots}^{j_m}$, for some $j_1 \neq \cdots \neq j_m$ belonging to $\{1, \ldots, M\}$, where $1 \leq m \leq M - 1$. Recall here the definition of $z'$ given in (2). For brevity, we use the shorthand notation $0^{j_1 \cdots j_m} := (0^{j_1})^{\cdots}^{j_m}$. Clearly, $0^{j_1 \cdots j_m} = 1$. As in the previous case, we consider existence and uniqueness of the optimum $\phi_i^{CDS}(t,z)$ implied by the first-order condition (24). This is done via a sequential procedure. Before proceeding further, we also introduce the notation $\hat{\phi}_i^{CDS}(t) := \hat{\phi}_i^{CDS}(t,0^{j_1 \cdots j_m})$. For $i = 1, \ldots, M$, we consider separate cases, using again the separable form $w^T(t,v,0^{j_1 \cdots j_m}) = v^T B^{(0^{j_1 \cdots j_m})}(t)$.

**Case 1.** If $i \neq j_1, \ldots, j_m$, then the first-order condition (24) for $z = 0^{j_1 \cdots j_m}$ is reduced to
\[
0 = B^{(0^{j_1 \cdots j_m})}(t)\left[\hat{\phi}_i^{L,t}(t,0^{j_1 \cdots j_m}) - L_i\right]h_i(0^{j_1 \cdots j_m})
- \sum_{l \neq i, l \neq j_1, \ldots, j_m} \left(\hat{\phi}_l^{L,t}(t,0^{j_1 \cdots j_m,l}) - \hat{\phi}_i^{L,t}(t,0^{j_1 \cdots j_m})\right)h_l(0^{j_1 \cdots j_m}).
\]
Then we obtain the following Lemma whose proof is reported in the Appendix.

Consequently, we obtain that Eq. (32) simplifies to

\[ \sum_{\ell \neq i, \neq j, \ldots, j_M - 1} \left( \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}}) - \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}}) \right) h^\gamma_i(0^{i1 \ldots J_{M-1}}) = 0, \]

and

\[ \sum_{\ell \neq i, \neq j, \ldots, j_M - 1} B^{(0^{i1 \ldots J_{M-1}})}(t) \left[ 1 + \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}}) \right]^{\gamma-1} \]

\[ \times h^\gamma_i(0^{i1 \ldots J_{M-1}}) \left( \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}}) - \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}}) \right) = 0. \]

Consequently, we obtain that Eq. (32) simplifies to

\[ 0 = B^{(0^{i1 \ldots J_{M-1}})}(t) \left( \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}}) - L_i \right) h_i(0^{i1 \ldots J_{M-1}}) \]

\[ + B^{(0^{i1 \ldots J_{M-1}})}(t) \left[ 1 + \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}}) \right]^{\gamma-1} \]

\[ \left( L_i - \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}}) \right) h^\gamma_i(0^{i1 \ldots J_{M-1}}), \]

where we used \( \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}}) = \hat{\phi}^{L,\nu}(t, 1) = L_i \), which leads, for \( i \neq j, \ldots, J_{M-1} \), to

\[ \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}}) = \frac{1}{L_i - \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}})} \left[ \left( \frac{B^{(0^{i1 \ldots J_{M-1}})}(t) h_i(0^{i1 \ldots J_{M-1}})}{B^{(0^{i1 \ldots J_{M-1}})}(t) h_i(0^{i1 \ldots J_{M-1}})} \right)^{\gamma-1} - 1 \right] \]

\[ = \frac{1}{L_i - \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}})} \left[ \left( \frac{B^{(0^{i1 \ldots J_{M-1}})}(t) h_i(0^{i1 \ldots J_{M-1}})}{B^{(0^{i1 \ldots J_{M-1}})}(t) h_i(0^{i1 \ldots J_{M-1}})} \right)^{\gamma-1} - 1 \right]. \]

**Case 2.** If \( j = j_k \) for some \( k = 1, \ldots, m \), then \( f^T_i(\phi, t, v, 0^{i1 \ldots J_{M-1}}) = 0 \). We have that \( \hat{\phi}^{CDS,*}_{1,1, \ldots, J_{M-1}}(t) = 0 \) in this case. Next, we distinguish again two subcases:

- **m = M - 1.** Then, it holds that for \( i \neq j, \ldots, J_{M-1} \),

\[ \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}}) = \hat{\phi}^{L,\nu}(t, 1) = L_i, \]

which leads, for \( i \neq j, \ldots, J_{M-1} \), to

\[ \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}}) = \frac{1}{L_i - \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}})} \left[ \left( \frac{B^{(0^{i1 \ldots J_{M-1}})}(t) h_i(0^{i1 \ldots J_{M-1}})}{B^{(0^{i1 \ldots J_{M-1}})}(t) h_i(0^{i1 \ldots J_{M-1}})} \right)^{\gamma-1} - 1 \right] \]

\[ = \frac{1}{L_i - \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}})} \left[ \left( \frac{B^{(0^{i1 \ldots J_{M-1}})}(t) h_i(0^{i1 \ldots J_{M-1}})}{B^{(0^{i1 \ldots J_{M-1}})}(t) h_i(0^{i1 \ldots J_{M-1}})} \right)^{\gamma-1} - 1 \right]. \]

- **m < M - 1.** Define the set

\[ N_{i,j_1, \ldots, j_{M-1}} = \{ \ell \in \{1, \ldots, M\}; \ell \neq i, \ell \neq j_1, \ldots, j_{M-1} \}. \]

Moreover, define

\[ B^{(j_1 \ldots j_m \ldots)}(t) := \left[ B^{(0^{i1 \ldots J_{m-1}})}(t), B^{(0^{i2 \ldots J_{m-1}})}(t), B^{(0^{i1 \ldots J_{m-1}})}(t); \ell \in N_{i,j_1, \ldots, j_{M-1}} \right]^T, \]

and the function on \( (t, B^{(j_1 \ldots j_m \ldots)}) \in [0, T] \times \mathbb{R}^{2+|N_{i,j_1, \ldots, j_{M-1}}|} \) by

\[ g_{j_1, \ldots, j_{M-1}} \left( \hat{\phi}^{CDS}_{1,1, \ldots, J_{M-1}}, B^{(j_1 \ldots j_m \ldots)} \right) := V_{j_1, \ldots, j_{M-1}}(t) B^{(j_1 \ldots j_m \ldots)}(t) \]

\[ + B^{(j_1 \ldots j_m \ldots)}(t) \left[ 1 + \hat{\phi}^{CDS}_{1,1, \ldots, J_{M-1}} \right] \left( L_i - \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}}) \right) \]

\[ + \sum_{\ell \in N_{i,j_1, \ldots, j_{M-1}}} B^{(j_1 \ldots j_m \ldots)} \left[ 1 + \hat{\phi}^{CDS}_{1,1, \ldots, J_{M-1}} \left( \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{m-1}}) - \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{m-1}}) \right) \right]^{\gamma-1} \]

\[ \times h^\gamma_i(0^{i1 \ldots J_{M-1}}) \left( \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{m-1}}) - \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{m-1}}) \right), \]

where the coefficient

\[ V_{j_1, \ldots, j_{M-1}}(t) := \left( \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{M-1}}) - L_i \right) h_i(0^{i1 \ldots J_{M-1}}) \]

\[ - \sum_{\ell \in N_{i,j_1, \ldots, j_{M-1}}} \left( \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{m-1}}) - \hat{\phi}^{L,\nu}(t, 0^{i1 \ldots J_{m-1}}) \right) h_i(0^{i1 \ldots J_{M-1}}). \]

Then we obtain the following Lemma whose proof is reported in the Appendix.
Lemma 4.2. There exists a unique optimum $\hat{z}_{i,j_1,\ldots,j_m}^{CDS,*}(t)$ satisfying the first-order condition (32) for each $i \neq j_1, \ldots, j_m$ subject to

$$M_1^{(i)}(t) < \hat{z}_{i,j_1,\ldots,j_m}^{CDS,*}(t) < M_2^{(i,m)}(t).$$

Moreover, the optimum $\hat{z}_{i,j_1,\ldots,j_m}^{CDS,*}(t)$, viewed as a function of $(t, B^{(j_1, \ldots, j_m,i)}) \in [0,T] \times \mathbb{R}^{2+|N_1,\ldots,N_m|}$, is continuous in $(t, B^{(j_1, \ldots, j_m,i)})$ for each $i = 1, \ldots, M$.

3. All names are defaulted, i.e. $z = 1$. Then the optimum $\hat{z}_{i,1}^{CDS}(t) = 0$.

4.2 Analysis of HJB Equation

We analyze the HJB equation (23). Using the separable form (26) of $w^T$, the HJB equation (23) may be rewritten as

$$0 = \frac{dB(t,z)}{dt} + \gamma \left( \sum_{i=1}^{M} \phi_i^{CDS,*}(1 - z_i) \left[ \left( \hat{\phi}_i^{L,t}(t,z) - L_i \right) h_i(z) \right. \right.$$

$$- \left. \left. \sum_{j \neq i} \left( \hat{\phi}_i^{L,t}(t,z_j) \right) (1 - z_j) h_j(z) \right] + [r + (\mu - r)\pi^{S,*}] \right) B(t,z)$$

$$+ \frac{1}{2} \gamma (\gamma - 1) \sigma^2 (\pi^{S,*})^2 B(t,z)$$

$$+ \sum_{i=1}^{M} \left[ \left( 1 + \phi_i^{CDS,*}(1 - z_i)(L_i - \hat{\phi}_i^{L,t}(t,z)) \right) \gamma B(t,z') - B(t,z) \right] (1 - z_j) h_j^0(z)$$

$$+ \sum_{i=1}^{M} \left[ \sum_{j \neq i} \left( 1 + \phi_i^{CDS,*}(1 - z_i)(\hat{\phi}_i^{L,t}(t,z_j) - \hat{\phi}_i^{L,t}(t,z)) \right) \gamma B(t,z') - B(t,z) \right] (1 - z_j) h_j^0(z).$$

(38)

In order to obtain an understanding of the structure of the above ODE, we first consider the case when $z = 0$. Then, we obtain

$$0 = \frac{dB^{(0)}(t,z)}{dt} + \gamma \left( \sum_{i=1}^{M} \phi_i^{CDS,*}(1 - z_i) \left[ \left( \hat{\phi}_i^{L,t}(t,0) - L_i \right) h_i(0) \right. \right.$$

$$- \left. \left. \sum_{j \neq i} \left( \hat{\phi}_i^{L,t}(t,0) \right) h_j(0) \right] B^{(0)}(t) + a(\gamma) B^{(0)}(t)$$

$$+ \sum_{i=1}^{M} \left[ \left( 1 + \phi_i^{CDS,*}(t)(L_i - \hat{\phi}_i^{L,t}(t,0)) \right) \gamma B^{(0)}(t) - B^{(0)}(t) \right] h_j^0(0)$$

$$+ \sum_{i=1}^{M} \left[ \sum_{j \neq i} \left( 1 + \phi_i^{CDS}(t)(\hat{\phi}_i^{L,t}(t,0) - \hat{\phi}_i^{L,t}(t,0)) \right) \gamma B^{(0)}(t) - B^{(0)}(t) \right] h_j^0(0).$$

(39)

The above equation indicates that $B^{(0)}(t)$, associated to the scenario where all names are alive, depends on $B^{(0)}(t)$, $j = 1, \ldots, M$, i.e. on the solutions of the ODEs associated to configurations where one entity defaults. This reflects the contagion mechanism in the control problem, and is a consequence of the interacting model governing the default intensities of the portfolio names. Iterating this procedure, it is immediate that we need to consider the equations associated to $B^{(0,\ldots,0,j_m+1)}(t)$ for all $j_1 \neq \cdots \neq j_m$ belonging to $\{1, \ldots, M\}$, where $1 \leq m \leq M$. Hence, from Eq. (38), we can establish the following inductive relationship between $B^{(0,\ldots,0,j_m+1)}(t)$ and $B^{(0,\ldots,0,j_m)}(t)$. For $1 \leq m \leq M - 1$,

$$0 = \frac{dB^{(0,\ldots,0,j_m)}(t,z)}{dt} + \gamma \left( \sum_{i \neq j, j_m} \phi_i^{CDS,*}(1 - z_i) \left[ \left( \hat{\phi}_i^{L,t}(t,0^{j_1,\ldots,j_m}) - L_i \right) h_i(0^{j_1,\ldots,j_m}) \right. \right.$$

$$- \left. \left. \sum_{j \neq i} \left( \hat{\phi}_i^{L,t}(t,0^{j_1,\ldots,j_m}) \right) h_j(0^{j_1,\ldots,j_m}) \right] B^{(0,\ldots,0,j_m)}(t)$$

$$+ \sum_{i=1}^{M} \left[ \left( 1 + \phi_i^{CDS,*}(t)(L_i - \hat{\phi}_i^{L,t}(t,0^{j_1,\ldots,j_m})) \right) \gamma B^{(0,\ldots,0,j_m)}(t) - B^{(0,\ldots,0,j_m)}(t) \right] h_j(0^{j_1,\ldots,j_m})$$

$$+ \sum_{i=1}^{M} \left[ \sum_{j \neq i} \left( 1 + \phi_i^{CDS}(t)(\hat{\phi}_i^{L,t}(t,0^{j_1,\ldots,j_m}) - \hat{\phi}_i^{L,t}(t,0^{j_1,\ldots,j_m})) \right) \gamma B^{(0,\ldots,0,j_m)}(t) - B^{(0,\ldots,0,j_m)}(t) \right] h_j(0^{j_1,\ldots,j_m}).$$

(37)
We next show via a backward procedure that a unique positive solution to Eq. (42) exists.

Hence Eq. (40) reduces to

\[
0 = \frac{d B^{(0^{1}, \ldots, 0^{j}, \ldots, 0^{m})}(t)}{dt} + \gamma \left\{ \sum_{\ell \neq i, \ell \neq j, \ldots, j_m} \frac{\vec{C}^{DS, *}_{1, j_1, \ldots, j_m}(t)}{L_i - \phi^L \nu_i(t, 0^{j_1}, \ldots, 0^{j_m})} B^{(0^{1}, \ldots, 0^{j}, \ldots, 0^{m})}(t) - B^{(0^{1}, \ldots, 0^{j}, \ldots, 0^{m})}(t) \right\} h_i^\nu (0^{j_1}, \ldots, 0^{j_m})
\]

The last term of Eq. (40) may be expanded further as

\[
\sum_{i=1}^{M} \left\{ \sum_{t \neq i, t \neq j_1, \ldots, j_m} \left[ \left( 1 + \frac{\vec{C}^{DS, *}_{1, j_1, \ldots, j_m}(t)(1 - z_i)\left( \phi^L \nu_i(t, 0^{j_1}, \ldots, 0^{j_m}) - \phi^L \nu_i(t, 0^{j_1}, \ldots, 0^{j_m}) \right) \right)^\gamma \times B^{(0^{1}, \ldots, 0^{j}, \ldots, 0^{m})}(t) - B^{(0^{1}, \ldots, 0^{j}, \ldots, 0^{m})}(t) \right] h_i^\nu (0^{j_1}, \ldots, 0^{j_m}) \right\} = \sum_{i=j_1, \ldots, j_m} \left\{ \sum_{t \neq i, t \neq j_1, \ldots, j_m} \left[ \left( 1 + \frac{\vec{C}^{DS, *}_{1, j_1, \ldots, j_m}(t)(1 - z_i)\left( \phi^L \nu_i(t, 0^{j_1}, \ldots, 0^{j_m}) - \phi^L \nu_i(t, 0^{j_1}, \ldots, 0^{j_m}) \right) \right)^\gamma \times B^{(0^{1}, \ldots, 0^{j}, \ldots, 0^{m})}(t) - B^{(0^{1}, \ldots, 0^{j}, \ldots, 0^{m})}(t) \right] h_i^\nu (0^{j_1}, \ldots, 0^{j_m}) \right\}
\]

Hence Eq. (40) reduces to

\[
0 = \frac{d B^{(0^{1}, \ldots, 0^{j}, \ldots, 0^{m})}(t)}{dt} + \gamma \left\{ \sum_{i \neq j_1, \ldots, j_m} \frac{\vec{C}^{DS, *}_{1, j_1, \ldots, j_m}(t)}{L_i - \phi^L \nu_i(t, 0^{j_1}, \ldots, 0^{j_m})} B^{(0^{1}, \ldots, 0^{j}, \ldots, 0^{m})}(t) - B^{(0^{1}, \ldots, 0^{j}, \ldots, 0^{m})}(t) \right\} h_i^\nu (0^{j_1}, \ldots, 0^{j_m})
\]

We next show via a backward procedure that a unique positive solution to Eq. (42) exists.
1. \( m = M \). This means that \( z = 1 \), hence \( B^{(1)}(t) \) satisfies
\[
\frac{dB^{(1)}(t)}{dt} = -a(\gamma) B^{(1)}(t), \quad B^{(1)}(T) = \frac{1}{\gamma}.
\]
yielding the positive solution
\[
B^{(1)}(t) = \frac{1}{\gamma} e^{a(\gamma)(T-t)}, \quad \text{for } 0 \leq t \leq T. \tag{43}
\]

2. \( m = M - 1 \). Then we have that \( \textbf{0}^{j_1, \ldots, j_{M-1}} = 1 \), for \( i \neq j_1, \ldots, j_{M-1} \). From Eq. (42), it follows that (setting \( j_M := \{1, \ldots, M\} \setminus \{j_1, \ldots, j_{M-1}\} \))
\[
0 = \frac{dB^{(0^{j_1, \ldots, j_{M-1}})}(t)}{dt} + a(\gamma) B^{(0^{j_1, \ldots, j_{M-1}})}(t)
+ \gamma \hat{\phi}_{j_M, j_1, \ldots, j_{M-1}} (\Phi_{L}^{(t)} (\textbf{0}^{j_1, \ldots, j_{M-1}}) - L_{j_M}) h_{j_M} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right) B^{(0^{j_1, \ldots, j_{M-1}})}(t)
+ \left[ \left( 1 + \gamma \hat{\phi}_{j_M, j_1, \ldots, j_{M-1}} (L_{j_M} - \Phi_{L}^{(t)} (\textbf{0}^{j_1, \ldots, j_{M-1}})) \right) \right] B^{(1)}(t) - B^{(0^{j_1, \ldots, j_{M-1}})}(t)
+ \sum_{i=1}^{j_1, \ldots, j_{M-1}} \left[ B^{(1)}(t) - B^{(0^{j_1, \ldots, j_{M-1}})}(t) \right] h_{j_M} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right). \tag{44}
\]
Substituting the optimum \( \hat{\phi}_{j_M, j_1, \ldots, j_{M-1}} \) given by (33) and the explicit solution \( B^{(1)}(t) \) given by (43) into (44), we conclude that Eq. (44) may be reduced to
\[
\frac{dB^{(0^{j_1, \ldots, j_{M-1}})}(t)}{dt} = D_{j_M} \left( t, B^{(0^{j_1, \ldots, j_{M-1}})}(t) \right), \quad B^{(0^{j_1, \ldots, j_{M-1}})}(T) = \frac{1}{\gamma}, \tag{45}
\]
where
\[
D_{j_M} (t, B) := -\left[ \gamma \hat{\phi}_{j_M, j_1, \ldots, j_{M-1}} (\Phi_{L}^{(t)} (\textbf{0}^{j_1, \ldots, j_{M-1}}) - L_{j_M}) h_{j_M} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right) + a(\gamma) \right] B
+ \left[ \left( 1 + \gamma \hat{\phi}_{j_M, j_1, \ldots, j_{M-1}} (L_{j_M} - \Phi_{L}^{(t)} (\textbf{0}^{j_1, \ldots, j_{M-1}})) \right) \right] B^{(1)}(t) - B^{(0^{j_1, \ldots, j_{M-1}})}(t)
- \sum_{i=1}^{j_1, \ldots, j_{M-1}} \left[ B^{(1)}(t) - B^{(0^{j_1, \ldots, j_{M-1}})}(t) \right] h_{j_M} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right),
= K_{j_M} B + U_{j_M} (t) B^{\frac{1}{\gamma}} + A_{j_M} (t),
\]
Above, the coefficients are given by
\[
K_{j_M} := M h_{j_M}^{\text{P}} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right) - \gamma h_{j_M} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right) - a(\gamma),
A_{j_M} (t) := -(M - 1) B^{(1)}(t) h_{j_M}^{\text{P}} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right),
U_{j_M} (t) := \gamma h_{j_M} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right) \left( \frac{h_{j_M} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right)}{B^{(1)}(t) h_{j_M}^{\text{P}} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right)} \right)^{\frac{1}{\gamma}}
- B^{(1)}(t) h_{j_M}^{\text{P}} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right) \left( \frac{h_{j_M} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right)}{B^{(1)}(t) h_{j_M}^{\text{P}} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right)} \right)^{\frac{1}{\gamma}}
= (\gamma - 1) h_{j_M} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right) \eta_{j_M} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right) B^{(1)}(t) t^{\frac{1}{\gamma}},
\]
with \( \eta_{j_M} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right) := \frac{h_{j_M} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right)}{h_{j_M}^{\text{P}} \left( \textbf{0}^{j_1, \ldots, j_{M-1}} \right)}. \)
Hence, we can conclude that the function \( B^{(0^{j_1, \ldots, j_{M-1}})}(t) \) satisfies the following ODE:
\[
u'(t) = K_{j_M} u(t) + U_{j_M} (t) u(t) t^{\frac{1}{\gamma}} + A_{j_M} (t), \quad u(T) = \frac{1}{\gamma},
\]
This implies that
\[
u(t) t^{\frac{1}{\gamma}} u'(t) = K_{j_M} u(t) t^{\frac{1}{\gamma}} + U_{j_M} (t) + A_{j_M} (t) u(t) t^{\frac{1}{\gamma}}.
\]
Let \( \tilde{u}(t) = u(t)^{1/\gamma} \). Then \( \tilde{u}'(t) = \frac{1}{\gamma} u(t)^{-1/\gamma} u'(t) \) and hence we have that

\[
\tilde{u}'(t) = K_{JM}(\gamma)\tilde{u}(t) + A_{JM}(t, \gamma)\tilde{u}(t)^{\gamma} + U_{JM}(t, \gamma), \quad \tilde{u}(T) = \gamma^{1/\gamma} > 0. \tag{46}
\]

where \( K_{JM}(\gamma) := \frac{K_{JM}}{\gamma^{1/\gamma}}, \ A_{JM}(t, \gamma) := \frac{A_{JM}(t)}{\gamma^{1/\gamma}} \) and \( U_{JM}(t, \gamma) := \frac{U_{JM}(t)}{\gamma^{1/\gamma}} \).

The ODE (46) is an inhomogeneous Bernoulli type ODE (also referred to as Chini’s equation). We then have the following Lemma whose proof is reported in the Appendix.

**Lemma 4.3.** There exists a unique positive solution to the ODE (46).

Recall that \( B^{(0_{1}, \ldots, j_{M-1})}(t)^{1/\gamma} = \tilde{u}(t) > 0 \) for all \( t \in [0, T] \). Then using Lemma 4.3, we deduce a unique positive solution \( B^{(0_{1}, \ldots, j_{M-1})}(t) \) to Eq. (45).

**Remark 4.4.** If \( \gamma = \frac{1}{2} \), then we have that \( \tilde{u}(t) = (B^{(0_{1}, \ldots, j_{M-1})}(t))^{2} > 0 \). Thus we have a unique positive solution to Eq. (45), which is given by \( B^{(0_{1}, \ldots, j_{M-1})}(t) = \sqrt{\tilde{u}(t)} \).

3. \( m \leq M - 2 \). We show existence and uniqueness of a positive solution \( B^{(0_{1}, \ldots, j_{M})}(t) \) to Eq. (42). Notice that, when \( m = 0 \), we have \( B^{(0_{1}, \ldots, j_{M})}(t) = B^{(0)}(t) \). We first give the following result.

**Lemma 4.5.** Let \( 0 \leq m < M - 1 \). Assume that there exists a unique positive solution \( B^{(0_{1}, \ldots, j_{M-1})}(t) \) to the ODE (42) in the domain \([0, T]\). Then, there exists a unique positive solution \( B^{(0_{1}, \ldots, j_{M})}(t) \) to (42) in the same domain.

The proof of this Lemma is reported in the Appendix. Recall that we know that a unique positive solution \( B^{(0_{1}, \ldots, j_{M-1})}(t) \) exists for any \( \{j_{1}, j_{2}, \ldots, j_{M-1}\} \subset \{1, \ldots, M\} \), from the previous case \( m = M - 1 \). Hence, applying Lemma 4.5, we obtain the existence of a unique positive solution \( B^{(0_{1}, \ldots, j_{M})}(t) \), for any \( \{j_{1}, j_{2}, \ldots, j_{M}\} \subset \{1, \ldots, M\} \). This concludes the analysis.

### 5 Main Results

We use the results derived in the previous section and show that the unique solution of the HJB equations coincide with the value function. Further, we identify the optimal admissible investment strategies in the stock and CDS assets.

Recall the operator defined by \( \mathcal{L} = \mathcal{L}_v + \mathcal{L}_s \) in Eq. (20). Let \( w(t, v, z) \) be \( C^1 \) in \( t \), and \( C^2 \) in \( v \). Then for \( t < u \),

\[
w(u, V_u, H(u)) = w(t, V_t, H(t)) + \int_t^u \left( \frac{\partial}{\partial s} + \mathcal{L} \right) w(s, V_s, H(s)) ds + \mathcal{M}_u - \mathcal{M}_t,
\]

where \( V = (V_t; t \geq 0) \) is the wealth process given in Lemma 3.1 and the \( \mathbb{P} \)-local martingale process is defined by

\[
\mathcal{M}_t := \int_0^t w(s, V_s, H(s)) \sigma V_s \pi^S_s dW_s + \sum_{i=1}^M \left\{ \int_0^t \left[ w \left( s, V_{s-} + V_{s-} \tilde{\Phi}_t^{CDS}(s-)(L_i - \tilde{\Phi}_t^{L^\nu}(t, H(s-))), H^\nu(s-) \right) - w(s, V_{s-}, H(s-)) \right] d\xi^S_s(s) \right\} + \sum_{j=1}^M \left\{ \sum_{j \neq j} \int_0^t \left[ w \left( s, V_{s-} + V_{s-} \tilde{\Phi}_t^{CDS}(s-)(L_j^{L^\nu}(s, H(s-))) - \tilde{\Phi}_t^{L^\nu}(t, H(s-)), H^\nu(s-) \right) - w(s, V_{s-}, H(s-)) \right] d\xi^S_j(s) \right\}, \quad t \geq 0. \tag{47}
\]

**Theorem 5.1.** For each \( m \in \{1, \ldots, M\} \), let \( j_k \in \{1, \ldots, M\} \) for all \( k = 1, \ldots, m \), where \( j_k \neq \ldots \neq j_m \).

Define \( \tilde{\Phi}_{t_{j_1}, \ldots, j_m}(t) \) as follows:
If \( i \neq j_1, \ldots, j_m \),

\[
\hat{\phi}^{CDS}_{i,j_1,\ldots,j_m}(t) := \begin{cases} 
\frac{1}{L_i} \phi_i^{L^\nu}(t,0^{i_1,\ldots,i_{m-1}}) \left\{ \frac{B(0^{i_1,\ldots,i_{m-1}})(t) h_i(0^{i_1,\ldots,i_{m-1}})}{B(1)(t) h_i(0^{i_1,\ldots,i_{m-1}})} \right\}^{1/\gamma} - 1, & m = M - 1; \\
0, & \text{satisfies the first-order condition (32),} \\
& 1 \leq m < M - 1; \\
& \text{satisfies the first-order condition (27),} \\
& m = 0,
\end{cases}
\]

where \( B(0^{i_1,\ldots,i_{m-1}})(t) \) is the unique positive solution to Eq. (44) and \( B(1)(t) = \frac{1}{\gamma} e^{\alpha(T-t)}, 0 \leq t \leq T. \)

If \( i = j_k \) for some \( k = 1, \ldots, m \), \( \hat{\phi}^{CDS}_{i,j_1,\ldots,j_m}(t) = 0. \)

Define

\[
\pi^{S,\ast}_i := \frac{1}{\gamma - 1} \frac{\mu - r}{\sigma^2}.
\]

Let \( B_{j_1,\ldots,j_m}^\ast(t) \) satisfy the ODE (42) when \( (\hat{\phi}^{CDS}_i(t,0^{i_1,\ldots,i_{m-1}}), \pi^S_i) = (\hat{\phi}^{CDS}_{i,j_1,\ldots,j_m}(t), \pi^{S,\ast}_i) \). Then the optimal fraction of wealth invested in the \( i \)-th CDS is given by \( \Phi^{L^\nu}_i(t,0^{i_1,\ldots,i_{m-1}}), \hat{\phi}^{CDS}_{i,j_1,\ldots,j_m}(t) \) and the optimal fraction of wealth invested in stock is \( \pi^{S,\ast}_i \). Moreover, the value function is given by

\[
w^T(t,v,0^{i_1,\ldots,i_{m-1}}) = v^\gamma B_{j_1,\ldots,j_m}^\ast(t), \quad \forall m = 1, \ldots, M.
\]

**Proof.** For \( z \in S \), define \( B(t,z) \) so that \( B(t,0^{i_1,\ldots,i_{m-1}}) := B(0^{i_1,\ldots,i_{m-1}})(t) \), where \( B(0^{i_1,\ldots,i_{m-1}})(t) \) satisfies the ODE (42). For any admissible feedback control: \( \Phi_i(t,z) = (\pi^S_i, \hat{\phi}^{CDS}_i(t,z); i = 1, \ldots, M) \), define the process

\[
Y_u^{\Phi} := \left( V_u(\Phi) \right)^\gamma B(u,H(u)), \quad u \geq t \geq 0.
\]

From Itô’s formula, it follows that

\[
Y_u^{\Phi} = Y_t^{\Phi} + \int_t^u F \left( \pi^S_i, \hat{\phi}^{CDS}_i(s,H(s)), \ldots, \hat{\phi}^{CDS}_m(s,H(s)); s, V_u^{\Phi}, H(s) \right) ds \\
+ \mathcal{M}^{\Phi}_u - \mathcal{M}^\Phi_t,
\]

where the \( \mathbb{P} \)-local martingale process is defined by

\[
\mathcal{M}^\Phi_t := \int_0^t \gamma \sigma \left( V_u(\Phi) \right)^\gamma B(s,H(s)) \pi^S_i dW_s \\
+ \sum_{i=1}^M \left\{ \int_0^t \left( V_u(\Phi) \right)^\gamma \left[ (1 + \hat{\phi}^{CDS}_i(s-,H(s-))(L_i - \Phi^{\nu}_i(t,H(s-))) \right)^\gamma B(s,H(t-)) \\
- B(s,H(s-)) \right] d\xi^\nu_i(s) \right\} \\
+ \sum_{i=1}^M \left\{ \sum_{j \neq i} \int_0^t \left( V_u(\Phi) \right)^\gamma \left[ (1 + \hat{\phi}^{CDS}_j(s-,H(s-))(\Phi^{\nu}_j(t,H(s-)) - \Phi^{\nu}_i(t,H(s-))) \right)^\gamma \\
\times B(s,H(s-)) - B(s,H(s-)) \right] d\xi^\nu_j(s) \right\}.
\]

while the function \( F(\Phi; t, v, z) \) is given by

\[
F(\Phi; t, v, z) = \gamma v^\gamma \left\{ \sum_{i=1}^M \hat{\phi}^{CDS}_i \left[ (\Phi^{\nu}_i(t,z) - L_i) h_i(z) - \sum_{j \neq i} (\Phi^{\nu}_j(t,z') - \Phi^{\nu}_i(t,z))(1 - z_j) h_j(z') \right] \right\} \\
\times B(t,z) + v^\gamma a (\pi^S_i, \gamma) B(t,z) \\
+ v^\gamma \frac{1}{M} \sum_{i=1}^M \left[ (1 + \hat{\phi}^{CDS}_i(1 - z_i)(L_i - \Phi^{\nu}_i(t,z)))^\gamma B(t,z') - B(t,z) \right](1 - z_i) h_i^\nu(z)
\]

(52)
\[ + v \gamma \sum_{i=1}^{M} \left\{ \sum_{j \neq i} \left[ (1 + \phi^{CDS}_{i}(1 - z_i)(\tilde{L}_{i}^{L,t}(t, z') - \tilde{L}_{i}^{R,t}(t, z,z)) \right] B(t, z') - B(t, z) \right\} (1 - z_j) \gamma h^3(z), \]

with \( a(\pi, \gamma) \) given by

\[ a(\pi, \gamma) := \gamma [r + (\mu - r)\pi] + \frac{1}{2} \gamma (1 - 1) \sigma^2 \pi^2. \]

Obviously, it holds that \( F_{\pi^S, \pi^S} = (\gamma (1 - 1)\sigma^2 v^\gamma B(t, z) \leq 0 \), while \( F_{\pi^S, \pi^S} = 0 \). Hence \( \pi^S \) given by \( \pi^S \) is the maximum of \( F \) w.r.t. \( \pi^S \). On the other hand, we have that for each \( i, j = 1, \ldots, M, \)

\[ F_{\tilde{\phi}^{CDS, \tilde{\phi}^{CDS}}_{i, j}} = \gamma (1 - 1)v^\gamma \left[ 1 + \phi^{CDS}_{i}(1 - z_i)(L_i - \phi^{CDS}_{1}(t, z)) \right] \gamma (L_i - \phi^{CDS}_{1}(t, z))^2 (1 - z_j) h^3(z) B(t, z^i) \]

\[ + \gamma (1 - 1)v^\gamma \left[ 1 + \phi^{CDS}_{i}(1 - z_i)(\phi^{CDS}_{1}(t, z') - \phi^{CDS}_{1}(t, z)^2) \right] \gamma (\phi^{CDS}_{1}(t, z') - \phi^{CDS}_{1}(t, z))^2 \]

\[ \leq 0, \]

\[ F_{\tilde{\phi}^{CDS, \tilde{\phi}^{CDS}}_{i, j}} = 0, \quad \forall i \neq j, \]

\[ F_{\tilde{\phi}^{CDS, \pi^S}_{i}} = 0, \quad \forall i = 1, \ldots, M. \]

This implies that the critical point is a maximum, and we denote it by \( \tilde{\phi}^*(t, z) \). Hence, the vector

\[ \tilde{\phi}^*(t, 0^{i_1 \cdots i_m}) = \left( \pi^S_i, \phi^{CDS}_{i_1 \cdots i_m}(t); \quad i = 1, \ldots, M \right) \]

is the optimum when \( z = 0^{i_1 \cdots i_m} \). Therefore, it holds that

\[ F(\tilde{\phi}; t, v, 0^{i_1 \cdots i_m}) \leq F(\tilde{\phi}^*(t, 0^{i_1 \cdots i_m}); t, v, 0^{i_1 \cdots i_m}) = 0, \]

where we have used that \( B(0^{i_1 \cdots i_m})(t) \) satisfies Eq. \( \cdot \), and \( B'(0^{i_1 \cdots i_m})(t) = B^*_{i_1 \cdots i_m}(t) \) when

\[ \left( \phi^{CDS}_{i}(t, 0^{i_1 \cdots i_m}), \pi^S_i \right) = \left( \phi^{CDS}_{i_1 \cdots i_m}(t), \pi^S_i \right). \]

Hence, for any admissible feedback control \( \tilde{\phi}(t, z) = (\pi^S_i, \phi^{CDS}_{i}(t, V_i(\tilde{\phi}), z); \quad i = 1, \ldots, M) \), it holds that for any \( 0 \leq t < < +\infty, \)

\[ \mathbb{E}^P_t \left[ Y_{\tilde{\phi}} \right] := \mathbb{E}^P_t \left[ Y_{\tilde{\phi}} \right] \leq \mathbb{E}^P_t \left[ Y_{\tilde{\phi}}^* + \mathbb{E}^P_t \left[ M_{\tilde{\phi}}^* - M_{\tilde{\phi}} \right] \right] \]

\[ = \left( V_i(\tilde{\phi}) \right)^\gamma B(t, H(t)) + \mathbb{E}^P_t \left[ M_{\tilde{\phi}}^* - M_{\tilde{\phi}} \right], \quad \text{(53)} \]

with equality when \( \tilde{\phi} = \tilde{\phi}^* \). Take \( u = T \wedge \tau_{a,b}, \) where \( \tau_{a,b} = \min \{ t \geq t; V_u \geq b^{-1} \text{or} \ V_u \leq a \} \) with \( 0 < a < V_0 = v < b \). Then

\[ \mathbb{E}^P_t \left[ Y_{\tilde{\phi}^*} \right] \leq \mathbb{E}^P_t \left[ V_{T \wedge \tau_{a,b}}(\tilde{\phi}^*) \right] \gamma B(T \wedge \tau_{a,b}, H(T \wedge \tau_{a,b})) \]

with the equality when \( \tilde{\phi} = \tilde{\phi}^* \). It remains to prove that

\[ \lim_{a,b \to 0} \mathbb{E}^P_t \left[ V_{T \wedge \tau_{a,b}}(\tilde{\phi}^*) \right] \gamma B(T \wedge \tau_{a,b}, H(T \wedge \tau_{a,b})) = \mathbb{E}^P_t \left[ U \left( V_{T \wedge \tau_{a,b}}(\tilde{\phi}^*) \right) \right]. \quad \text{(54)} \]

From the continuous differentiability of the function \( B(t, 0^{i_1 \cdots i_m}) \) w.r.t. \( t \in [0, T] \), it follows that there exist positive constants \( c_1 \) and \( c_2 \) so that

\[ \mathbb{E}^P_t \left[ \left( V_{T \wedge \tau_{a,b}}(\tilde{\phi}^*) \right) \gamma B(T \wedge \tau_{a,b}, H(T \wedge \tau_{a,b})) \right] \leq c_1 + c_2 \mathbb{E}^P_t \left[ V_{T \wedge \tau_{a,b}}(\tilde{\phi}^*) \right]^2. \quad \text{(55)} \]

Define

\[ \alpha(t, v, z) := v \left\{ r + (\mu - r)\pi^S_i + \sum_{i=1}^{M} \phi^{CDS, \pi^S}_{i}(t, z, B(t, z))(1 - z_i) \left[ (\tilde{L}_{i}^{L,t}(t, z) - L_i)(h_i(z) - h^3(z)) \right] \right\} \]

\[ + \sum_{j \neq i} \left[ (\tilde{L}_{i}^{L,t}(t, z') - \tilde{L}_{i}^{R,t}(t, z))(1 - z_j) (h^3(z) - h_j(z)) \right]. \]

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\[ \beta_i(t, v, z) := \psi^{\text{CDS}^*}_i(t, z, B(t, z))(1 - z_i) \left( L_i - \hat{\Phi}_i^{L, u}(t, z) \right), \]
\[ \theta_{ij}(t, v, z) := \psi^{\text{CDS}^*}_i(t, z, B(t, z))(1 - z_i) \left( \hat{\Phi}_i^{L, u}(t, z') - \hat{\Phi}_i^{L, u}(t, z) \right)(1 - z_j), \quad j \neq i, \]
\[ \vartheta(t, v) := \psi^S_i(t, v). \]

Then the P-dynamics of the wealth process \( V_t(\hat{\phi}^*) \) may be rewritten as
\[
dV_t(\hat{\phi}^*) = \alpha(t, V_t(\hat{\phi}^*), H(t))dt + \sum_{i=1}^{M} \beta_i(t, V_t(\hat{\phi}^*), H(t))d\xi^P_i(t) + \vartheta(t, V_t(\hat{\phi}^*))dW_t.
\]

By (37), we have that for each \( i, m = 1, \ldots, M, M_1^{(i)}(t) < \phi^{\text{CDS}^*}_i(t) < M_2^{(i,m)}(t) \) where \( t \in [0, T] \). Next we assume that \( M_2^{(i,m)}(t) < +\infty \). Let \( c_{M,T} \) be a generic constant depending on \( M \) and \( T \) that may be different for each inequality below. For \( 0 \leq t < u \leq T \), using Hölder’s inequality
\[
\left| \int_t^u \alpha \left( s, V_s(\hat{\phi}^*), H(s) \right) ds \right|^2 \leq (T - t)c_{M,T} \int_t^u \left( V_s(\hat{\phi}^*) - V_t(\hat{\phi}^*) \right)^2 ds \leq c_{M,T} \left[ (T - t) \left| V_t(\hat{\phi}^*) \right|^2 + \int_t^u \left| V_s(\hat{\phi}^*) - V_t(\hat{\phi}^*) \right|^2 ds \right].
\]

From B-D-G inequality (see, e.g., Karatzas and Shreve (1991)), it follows that
\[
\mathbb{E}_t^p \left[ \sup_{t \leq s \leq T} \left| \sum_{i=1}^{M} \int_t^u \beta_i \left( s, V_s(\hat{\phi}^*), H(s) \right) d\xi^P_i(s) \right|^2 \right] \leq c_{M,T} \sum_{i=1}^{M} \mathbb{E}_t^p \left[ \int_t^u \beta_i \left( s, V_s(\hat{\phi}^*), H(s) \right) d\xi^P_i(s) \right]^2 \leq c_{M,T} \sum_{i=1}^{M} \mathbb{E}_t^p \left[ \int_t^T \beta_i \left( s, V_s(\hat{\phi}^*), H(s) \right) dH_i(s) \right] \leq c_{M,T} \mathbb{E}_t^p \left[ (T - t) \left| V_t(\hat{\phi}^*) \right|^2 + \int_t^u \left| V_s(\hat{\phi}^*) - V_t(\hat{\phi}^*) \right|^2 ds \right].
\]

Similarly, it holds that
\[
\mathbb{E}_t^p \left[ \sup_{t \leq s \leq T} \left| \sum_{i=1}^{M} \sum_{j \neq i} \int_t^u \theta_{ij} \left( s, V_s(\hat{\phi}^*), H(s) \right) d\xi^P_i(s) \right|^2 \right] \leq c_{M,T} \mathbb{E}_t^p \left[ (T - t) \left| V_t(\hat{\phi}^*) \right|^2 + \int_t^u \left| V_s(\hat{\phi}^*) - V_t(\hat{\phi}^*) \right|^2 ds \right].
\]

Using B-D-G inequality again, we have that
\[
\mathbb{E}_t^p \left[ \sup_{t \leq s \leq T} \left| \int_t^u \vartheta \left( s, V_s(\hat{\phi}^*) \right) dW_s \right|^2 \right] \leq c_{M,T} \mathbb{E}_t^p \left[ (T - t) \left| V_t(\hat{\phi}^*) \right|^2 + \int_t^u \left| V_s(\hat{\phi}^*) - V_t(\hat{\phi}^*) \right|^2 ds \right].
\]

Thus by Gronwall’s inequality, we can conclude that the wealth process satisfies the moment condition:
\[
\mathbb{E}_t^p \left[ \sup_{t \leq s \leq T} \left| V_s(\hat{\phi}^*) - V_t(\hat{\phi}^*) \right|^2 \right] \leq c_{M,T} \left| V_t(\hat{\phi}^*) \right|^2 + d_{M,T}, \tag{56}
\]
where \( d_M \) is also a positive constants depending on \( M \) and on the terminal time \( T \).

As for the case where \( M_2^{(i,m)}(t) < +\infty \), notice that
\[
\hat{\phi}_i^{\text{CDS}^*}(t, 0^{j_1, \ldots, j_m}) = \hat{\phi}_i^{\text{CDS}^*}(t, B(0^{j_1, \ldots, j_m})),
\]
for all \( i \neq j_1 \neq \ldots \neq j_m \), and
\[
B(0^{j_1, \ldots, j_m}) = \left[ B(t, 0^{j_1, \ldots, j_m}), B(t, 0^{j_1, \ldots, j_m, \ell}) ; \ell \in N_{j_1, j_2, \ldots, j_m} \right]^T.
\]

Since \( t \in [0, T] \), which is a bounded closed set, we have that for any \( i, m \in \{1, \ldots, m\} \) and \( i \neq j_1 \neq \ldots \neq j_m \), the optimum \( \hat{\phi}_i^{\text{CDS}^*}(t, 0^{j_1, \ldots, j_m}, B(t, 0^{j_1, \ldots, j_m})) \) is bounded using the continuity of \( \hat{\phi}_i^{\text{CDS}^*}(t, B^{(j_1, \ldots, j_m)}) \).
6.1 Investor Strategies and Value Functions

and provide more explicit characterizations for the optimal value functions and CDS strategies. Section 6.2
assesses the behavior of investor strategies in case when

\[ \sup_{0 < a < v < b < \infty} E\left[ \left| V_{T_{\tau a, b}}(\hat{\phi}^*) \right|^2 \right] \leq 2 \left| V_i(\hat{\phi}^*) \right|^2 + 2E\left[ \sup_{t \leq u \leq T} \left| V_u(\hat{\phi}^*) - V_i(\hat{\phi}^*) \right|^2 \right] < +\infty. \]

Then, the equality (54) follows from Corollary 7.15 in Chow and Teicher (1978). This completes the proof of
the theorem.

6 Numerical Analysis

We assess the behavior of investor strategies in case when \( M = 2 \). Section 6.1 specializes the framework
and provide more explicit characterizations for the optimal value functions and CDS strategies. Section 6.2
illustrates our economic findings through a comparative statics analysis.

6.1 Investor Strategies and Value Functions

When \( M = 2 \), we have the first-order derivative functions \( f_1^{T'}(\phi; t, v, z) \) and \( f_2^{T'}(\phi; t, v, z) \) given by

\[
f_1^{T'}(\phi; t, v, z) = vw_T(t, v, z)(1 - z_1) \left[ \left( \hat{\theta}_1^{L,\nu}(t, z_1, z_2) - \hat{\theta}_1^{L,\nu}(t, 1 - z_1, z_2) \right) h_1(z) \right. \\
- \left. \left( \hat{\theta}_1^{L,\nu}(t, z_1, 1 - z_2) - \hat{\theta}_1^{L,\nu}(t, z_1, z_2) \right) (1 - z_2) h_2(z) \right] \\
+ vw_T(t, v + \nu CDS (1 - z_1)(L_1 - \hat{\theta}_1^{L,\nu}(t, z_1, z_2)), (1 - z_1, z_2)) (L_1 - \hat{\theta}_1^{L,\nu}(t, z_1, z_2))(1 - z_1)h_0(z) \\
+ vw_T(t, v + \nu CDS (1 - z_1)) (\hat{\theta}_1^{L,\nu}(t, z_1, 1 - z_2) - \hat{\theta}_1^{L,\nu}(t, z_1, z_2)), (z_1, 1 - z_2)) \\
\times (1 - z_2)h_2(z)(1 - z_1) \left( \hat{\theta}_1^{L,\nu}(t, z_1, 1 - z_2) - \hat{\theta}_1^{L,\nu}(t, z_1, z_2) \right),
\]

and

\[
f_2^{T'}(\phi; t, v, z) = vw_T(t, v, z)(1 - z_2) \left[ \left( \hat{\theta}_2^{L,\nu}(t, z_1, z_2) - \hat{\theta}_2^{L,\nu}(t, 1 - z_1, z_2) \right) h_1(z) \right. \\
- \left. \left( \hat{\theta}_2^{L,\nu}(t, 1 - z_1, z_2) - \hat{\theta}_2^{L,\nu}(t, z_1, z_2) \right) (1 - z_2) h_1(z) \right] \\
+ vw_T(t, v + \nu CDS (1 - z_2)(L_2 - \hat{\theta}_2^{L,\nu}(t, z_1, z_2)), (z_1, 1 - z_2)) \\
\times (L_2 - \hat{\theta}_2^{L,\nu}(t, z_1, z_2))(1 - z_2)h_0(z) \\
+ vw_T(t, v + \nu CDS (1 - z_2)) (\hat{\theta}_2^{L,\nu}(t, 1 - z_1, z_2) - \hat{\theta}_2^{L,\nu}(t, z_1, z_2)), (1 - z_1, z_2)) \\
\times (1 - z_1)h_1(z)(1 - z_2) \left( \hat{\theta}_2^{L,\nu}(t, 1 - z_1, z_2) - \hat{\theta}_2^{L,\nu}(t, z_1, z_2) \right).
\]

For convenience, we write \( w^{T,(i,j)}(t, v) := w^T(t, v, (i, j)) \), where \((i, j) \in \{0, 1\}^2 \). Then we have the following cases:

- For \( z = (z_1, z_2) = (0, 0) \), we have that

\[
f_1^{T'}(\phi; t, v, z) = vw_T^{(0,0)}(t, v) \left[ \left( \hat{\theta}_1^{L,\nu}(t, 0, 0) - \hat{\theta}_1^{L,\nu}(t, 1, 0) \right) h_1(0, 0) \right. \\
- \left. \left( \hat{\theta}_1^{L,\nu}(t, 0, 1) - \hat{\theta}_1^{L,\nu}(t, 0, 0) \right) h_2(0, 0) \right] \\
+ vw_T^{(1,0)} \left( t, v + \nu CDS (L_1 - \hat{\theta}_1^{L,\nu}(t, 0, 0)) \right) (L_1 - \hat{\theta}_1^{L,\nu}(t, 0, 0))h_0(0, 0) \\
+ vw_T^{(0,1)} \left( t, v + \nu CDS (\hat{\theta}_1^{L,\nu}(t, 0, 1) - \hat{\theta}_1^{L,\nu}(t, 0, 0)) \right) \\
\times \left( \hat{\theta}_1^{L,\nu}(t, 0, 1) - \hat{\theta}_1^{L,\nu}(t, 0, 0) \right) h_2(0, 0),
\]
and
\[
f^{T,i}_{2}(\phi; t, v, z) = vw^{T,(0)}_{v}(t, v) \left[ \left( \tilde{\Phi}^{L,v}_{2}(t, 0, 0) - \tilde{\Phi}^{L,v}_{2}(t, 0, 1) \right) h_{2}(0, 0) - \left( \tilde{\Phi}^{L,v}_{2}(t, 1, 0) - \tilde{\Phi}^{L,v}_{2}(t, 0, 0) \right) h_{1}(0, 0) \right]
\]
\[
+ \left[ \left( \tilde{\Phi}^{L,v}_{2}(t, 0, 0) - \tilde{\Phi}^{L,v}_{2}(t, 0, 1) \right) h_{2}(0, 0) - \left( \tilde{\Phi}^{L,v}_{2}(t, 1, 0) - \tilde{\Phi}^{L,v}_{2}(t, 0, 0) \right) h_{1}(0, 0) \right]
\]
\[
+ \left( \tilde{\Phi}^{L,v}_{2}(t, 1, 0) - \tilde{\Phi}^{L,v}_{2}(t, 0, 0) \right) h_{1}(0, 0).
\]

- For \( z = (z_{1}, z_{2}) = (1, 0) \), we have that \( f^{T,i}_{1}(\phi; t, v, z) = 0 \), and
\[
f^{T,i}_{1}(\phi; t, v, z) = vw^{T,(0,1)}_{v}(t, v) \left( \tilde{\Phi}^{L,v}_{1}(t, 0, 0) - L_{2} \right) \left( h_{2}(0, 0) \right)
\]
\[
+ \left( \tilde{\Phi}^{L,v}_{1}(t, 0, 0) - L_{2} \right) \left( h_{2}(0, 0) \right) \left( \tilde{\Phi}^{L,v}_{1}(t, 0, 0) - L_{2} \right) \left( h_{2}(0, 0) \right)
\]
\[
+ \left( \tilde{\Phi}^{L,v}_{1}(t, 0, 0) - L_{2} \right) \left( h_{2}(0, 0) \right) \left( \tilde{\Phi}^{L,v}_{1}(t, 0, 0) - L_{2} \right) \left( h_{2}(0, 0) \right)
\]
\[
\times \left( \tilde{\Phi}^{L,v}_{1}(t, 1, 0) - \tilde{\Phi}^{L,v}_{1}(t, 0, 0) \right) h_{1}(0, 0).
\]

Consider the form (26) of the value function \( w^{T} \) and use the simplified notation \( B^{(x)}(t) = B(t, z) \). Note that \( \tilde{\Phi}^{L,v}_{1}(t, 1, 1) = L_{1} \) and \( \tilde{\Phi}^{L,v}_{1}(t, 1, 1) = L_{2} \) using Lemma A.2. Then we have that

**Lemma 6.1.** The optimum \( \hat{\phi}^{CDS,*}_{1} = (\hat{\phi}^{CDS,*}_{1}, \hat{\phi}^{CDS,*}_{2}) \) on CDS is given by

- \( \hat{\phi}^{CDS,*}_{1}(t, 1, 1) = \hat{\phi}^{CDS,*}_{2}(t, 1, 1) = 0; \)
- \( \hat{\phi}^{CDS,*}_{1}(t, 1, 0) = \hat{\phi}^{CDS,*}_{2}(t, 0, 1) = 0; \)
- We have
\[
\hat{\phi}^{CDS,*}_{1}(t, 0, 1) = \frac{1}{L_{1} - \tilde{\Phi}^{L,v}_{1}(t, 0, 1)} \left\{ \left[ B^{(0,1)}(t)(L_{1} - \tilde{\Phi}^{L,v}_{1}(t, 0, 1)) h_{1}(0, 1) \right] \frac{1}{B^{(1,1)}(t)(L_{1} - \tilde{\Phi}^{L,v}_{1}(t, 0, 1)) h_{1}(0, 1)} - 1 \right\},
\]
and
\[
\hat{\phi}^{CDS,*}_{2}(t, 1, 0) = \frac{1}{L_{2} - \tilde{\Phi}^{L,v}_{2}(t, 1, 0)} \left\{ \left[ B^{(1,0)}(t)(L_{2} - \tilde{\Phi}^{L,v}_{2}(t, 1, 0)) h_{2}(0, 1) \right] \frac{1}{B^{(1,1)}(t)(L_{2} - \tilde{\Phi}^{L,v}_{2}(t, 1, 0)) h_{2}(0, 1)} - 1 \right\}.
\]

Moreover, \( \hat{\phi}^{CDS,*}_{1}(t, 0, 0) \) satisfies the equation:
\[
0 = V_{1}(t)B^{(0,0)}(t) + B^{(1,0)}(t)(L_{1} - \tilde{\Phi}^{L,v}_{1}(t, 0, 0)) h_{1}(0, 0) \left[ 1 + \hat{\phi}^{CDS,*}_{1}(t, 0, 0)(L_{1} - \tilde{\Phi}^{L,v}_{1}(t, 0, 0)) \right]^{-1}
\]
\[
+ B^{(0,1)}(t) \left( \tilde{\Phi}^{L,v}_{1}(t, 0, 1) - \tilde{\Phi}^{L,v}_{1}(t, 0, 0) \right) h_{1}(0, 0)
\]
\[
\times \left[ 1 + \hat{\phi}^{CDS,*}_{1}(t, 0, 0) \left( \tilde{\Phi}^{L,v}_{1}(t, 0, 1) - \tilde{\Phi}^{L,v}_{1}(t, 0, 0) \right) \right]^{-1},
\]
and \( \hat{\phi}^{CDS,*}_{2}(t, 0, 0) \) satisfies the equation:
\[
0 = V_{2}(t)B^{(0,0)}(t) + B^{(0,1)}(t)(L_{2} - \tilde{\Phi}^{L,v}_{2}(t, 0, 0)) h_{2}(0, 0) \left[ 1 + \hat{\phi}^{CDS,*}_{2}(t, 0, 0)(L_{2} - \tilde{\Phi}^{L,v}_{2}(t, 0, 0)) \right]^{-1}
\]
\[
\times \left[ 1 + \hat{\phi}^{CDS,*}_{2}(t, 0, 0) \left( \tilde{\Phi}^{L,v}_{2}(t, 0, 1) - \tilde{\Phi}^{L,v}_{2}(t, 0, 0) \right) \right]^{-1}.
\]
\[ +B^{(1,0)}(t) \left( \hat{\varphi}_2^{L,\nu}(t, 1, 0) - \hat{\varphi}_2^{L,\nu}(t, 0, 0) \right) h_1^2(0, 0) \]
\[ \times \left[ 1 + \hat{\varphi}_2^{CDS,*}(t, 0, 0) \left( \hat{\varphi}_2^{L,\nu}(t, 1, 0) - \hat{\varphi}_2^{L,\nu}(t, 0, 0) \right) \right]^{\gamma-1}, \]  

(60)

where

\[ V_1(t) := \left( \hat{\varphi}_1^{L,\nu}(t, 0, 0) - L_1 \right) h_1(0, 0) - \left( \hat{\varphi}_1^{L,\nu}(t, 0, 1) - \hat{\varphi}_1^{L,\nu}(t, 0, 0) \right) h_2(0, 0), \]
\[ V_2(t) := \left( \hat{\varphi}_2^{L,\nu}(t, 0, 0) - L_2 \right) h_2(0, 0) - \left( \hat{\varphi}_2^{L,\nu}(t, 1, 0) - \hat{\varphi}_2^{L,\nu}(t, 0, 0) \right) h_1(0, 0). \]

**Proof.** The above expressions for \( \hat{\varphi}_i^{CDS,*}(t, z), i = 1, 2 \) and \( z \neq 0 \), come from a direct calculation. Equations (59) and (60) are derived from the first-order condition (24) using the expressions for \( f_1^{T^r} \) and \( f_2^{T^r} \) given above. \( \square \)

In what follows, we define the following vectors

\[ \mathbf{B}^{(1)}(t) := \left[ B^{(0,0)}(t), B^{(1,0)}(t), B^{(0,1)}(t) \right]^T, \]

and

\[ \mathbf{B}^{(2)}(t) := \left[ B^{(0,0)}(t), B^{(0,1)}(t), B^{(1,0)}(t) \right]^T. \]

As a corollary of Lemma 4.1 for the case \( M \geq 2 \), we have that

**Corollary 6.2.** There exists a unique optimum \( \hat{\varphi}_i^{CDS,*}(t, 0, 0) \) satisfying the first-order conditions (59) and (60). Moreover, for each \( i = 1, 2 \), the optimum \( \hat{\varphi}_i^{CDS,*}(t, 0, 0) \) viewed as function of \((t, \mathbf{B}^{(i)}) \in [0, T] \times \mathbb{R}_+^2\) is continuous in \((t, \mathbf{B}^{(i)})\) for each \( i = 1, 2 \).

Next, we provide the HJB equations when \( M = 2 \). Recall the notation \( B^{(2)}(t) := B(t, z) \) for each \( z \in \{0, 1\}^2 \). Then, we have

**Lemma 6.3.** The positive functions \( B^{(1,1)}(t), B^{(1,0)}(t), B^{(0,1)}(t) \) and \( B^{(0,0)}(t) \) satisfy the following equations:

\[ 0 = \frac{dB^{(1,1)}(t)}{dt} + \gamma \left[ r + (\mu - r)\pi^{S,*} \right] + \frac{1}{2} (\gamma - 1) \sigma^2 (\pi^{S,*})^2 \right] B^{(1,1)}(t), \]

(61)

\[ 0 = \frac{dB^{(1,0)}(t)}{dt} + \gamma \left[ \hat{\varphi}_2^{CDS,*}(t, 1, 0) \left( \hat{\varphi}_2^{L,\nu}(t, 1, 0) - L_2 \right) h_2(1, 0) + [r + (\mu - r)\pi^{S,*}] \right] B^{(1,0)}(t) \]

\[ + \frac{1}{2} \gamma (\gamma - 1) \sigma^2 (\pi^{S,*})^2 B^{(1,0)}(t) \]

(62)

\[ + \left[ \left( 1 + \hat{\varphi}_2^{CDS,*}(t, 0, 0) (L_2 - \hat{\varphi}_2^{L,\nu}(t, 1, 0)) \right)^\gamma \right] B^{(1,1)}(t) - B^{(1,0)}(t) \]

\[ + \left[ B^{(1,1)}(t) - B^{(1,0)}(t) \right] h_2^2(1, 0), \]

\[ 0 = \frac{dB^{(0,1)}(t)}{dt} + \gamma \left[ \hat{\varphi}_1^{CDS,*}(t, 0, 1) + (\hat{\varphi}_1^{L,\nu}(t, 0, 1) - L_1) h_1(0, 1) + [r + (\mu - r)\pi^{S,*}] \right] B^{(0,1)}(t) \]

\[ + \frac{1}{2} \gamma (\gamma - 1) \sigma^2 (\pi^{S,*})^2 B^{(0,1)}(t) \]

(63)

\[ + \left[ \left( 1 + \hat{\varphi}_1^{CDS,*}(t, 0, 1) (L_1 - \hat{\varphi}_1^{L,\nu}(t, 0, 1)) \right)^\gamma \right] B^{(1,1)}(t) - B^{(0,1)}(t) \]

\[ + \left[ B^{(1,1)}(t) - B^{(0,1)}(t) \right] h_1^2(0, 1), \]

and

\[ 0 = \frac{dB^{(0,0)}(t)}{dt} + \gamma \left[ \sum_{i=1}^2 \hat{\varphi}_i^{CDS,*}(t, 0, 0) \left( \hat{\varphi}_i^{L,\nu}(t, 0, 0) - L_i \right) h_i(0, 0) \right] \]

(64)
of her strategy, moving from long credit position (pays LGD) to a short credit position (pays the spread 1, we observe that as the default intensity of name “2” gets larger, the investor changes the directionality effect becomes more pronounced in case of the CDS referencing name “1”. From the top right panel of Figure “2”. Consequently, the risk averse investor will allocate a smaller proportion of wealth to it. The contagion the compensation for bearing default risk remains the same as time progresses. Consequently, the investor

larger. This is justified by the fact that risk-premium, historical and risk neutral default intensities are increase (jumping upwards from \( \nu_2 \)). This happens because at the default of name “1”, the default intensity of name “2” will instantaneously default intensity of name “1” increases, the investor decreases the proportion of wealth allocated to CDS also illustrate how default contagion impacts the investment strategy. It appears from Figure 2 that, as the

in default of “1” increases. On other hand, if name “1” has already defaulted the investor strategy in CDS “2” will slightly reduces the amount of units purchased as the default intensity reached by name “2” after default (hence the protection leg has a higher value than the premium leg), and buys it when it is negative. They

be more sensitive to increases in default intensity of name “2”. This is because the investor is now certain of “1” increases. On other hand, if name “1” has already defaulted the investor strategy in CDS “2” will

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Figure 1: The top panels report the dependence of the optimal investment strategy in the CDS referencing name “1” on the default intensity of name “1” and “2” before any default happens. The bottom panels report the dependence of the CDS market value on both intensities. We fix $h_{P1}(0,1) = h_{P2}(1,0) = 2.3$. Whenever varying $h_{P2}(0,0)$ ($h_{P1}(0,0)$), we fix $h_{P1}(0,0) = 0.2$ ($h_{P2}(0,0) = 0.2$).
Figure 2: The top panels report the dependence of the optimal investment strategy in the CDS referencing name “2” on the default intensity of name “1” and “2” before any default happens. The bottom panels report the dependence of the CDS market value on both intensities. We fix $h_{P1}(0,0) = h_{P2}(1,0) = 2.3$. Whenever varying $h_{P2}(0,0)$ ($h_{P1}(0,0)$), we fix $h_{P1}(0,0) = 0.2$ ($h_{P2}(0,0) = 0.2$).
Figure 3: The top left panel reports the dependence of the optimal CDS '2' investment strategy on changes in the default intensity of name ‘2’ after name ‘1’ defaults. The top right panel shows the same dependence, but under the scenario when ‘1’ has already defaulted. The bottom panels report the dependence of the corresponding CDS market values on the corresponding default intensities. We fix $h_{1}^{P}(0,0) = h_{2}^{P}(0,0) = 0.2$. Whenever varying $h_{2}^{P}(0,0)$ ($h_{1}^{P}(0,0)$), we fix $h_{1}^{P}(0,1) = 2.3$ ($h_{2}^{P}(1,0) = 2.3$).
Figure 4: The graphs describe the dependence of the pre-default CDS investment strategy on the joint behavior of the pre-default intensities of both names. We fix $t = 0$, $h_{11}(0,1) = h_{22}(1,0) = 2.3$.

Figure 5: The graphs describe the dependence of the optimal CDS investment strategy on the price differences before and after default. We fix $t = 0$, $h_{11}(0,0) = h_{22}(0,0) = 0.2$.

should buy more (sell less) units of CDS when the market valuation of CDS is low (high), that is for longer time to maturity, all else being equal. Similar findings are also obtained from Bielecki and Jang (2006) and by Capponi and Lopez (2012), but in a different framework.

Next, we investigate how the investment in CDS “1” and “2” depends on the joint behavior of the pre-default intensities. The graphs in Figure 4 confirm how the contagion effect generated by the default of name “2” (respectively name “1”) affects the proportion of wealth invested in CDS “1” (respectively CDS “2”). When the default intensity of name “2” is high, the contagion effect on name “1” is stronger, hence the investment strategy in CDS “1” has a smaller sensitivity to changes in the pre-default intensity of name “1”. This occurs because default of name “2” is very likely to occur, and hence the post-default intensity of name “1” has a larger impact than the pre-default intensity on the market valuation of CDS “1”. Viceversa, if the default intensity of name “2” is small, the contagion effect is small, and the investment strategy in CDS “1” mostly depends on the pre-default intensity of name “1”. Similar patterns occur for the investment strategy in the CDS referencing name “2”, as indicated from the bottom right graph.

We conclude with an illustration of how the optimal proportion of wealth invested in CDSs changes with respect to the price difference of a credit default swap after and before the default of the other name in the portfolio. As expected, when name “2” defaults the market value of the CDS referencing name “1” increases due to the increased default intensity of “1”. As a result, the investor would then sell CDS units referencing “1”. Figure 5 shows that as the difference between the price of CDS “1” after and before the default of “2” gets larger, the investor sells an increasingly higher number of CDS “1” units. The top right panel of Figure 5 confirms a similar trend for CDS “2”.

26
7 Conclusions

We have presented a novel portfolio optimization framework to study the effects generated by default contagion on the optimal investment strategies. We have considered a power investor who can allocate her wealth across a stock index, multiple credit default swaps, and a money market account. Using a Markovian interacting intensity framework, we have derived the optimal strategies using the HJB approach. We have provided a rigorous analysis showing that the value functions can be characterized as the unique positive solutions of interrelated inhomogeneous Bernoulli’s type ODEs. Such a dependence reflects the interacting nature of the default process, and establishes a lattice dependence structure. In such a structure, the value function associated to the scenario configuration where all names are alive acts as the minimum, while the one associated to the scenario configuration where all names are alive acts as the minimum. By means of a numerical analysis, we have demonstrated that default contagion has a strong effect on the allocation decisions of the investor, pushing him to sell higher number of CDS units when the CDS market valuation increases as the result of a contagious default event.

A Proofs of Theorems, Lemmas, and Propositions

Proof of Lemma 2.1.

Proof. It is enough to prove that, for every nonnegative predictable process \(Y_t\), we have

\[
\mathbb{E} \left[ \int_0^T Y_t dH_i(t) \right] = \mathbb{E} \left[ \int_0^T Y_t(1 - H_i(t))h_i(H(t))dt \right], \quad \text{for any } T > 0,
\]

(A.1)

where we recall that \(\mathbb{E}\) denotes the expectation w.r.t. the probability measure \(\mathbb{Q}\), and \(h_i(H(t))\) the risk-neutral default intensity.

Consider the right hand side of Eq. (A.1). Then, we obtain

\[
\mathbb{E} \left[ \int_0^T Y_t(1 - H_i(t))h_i(H(t))dt \right] = \mathbb{E}^P \left[ \int_0^T X_t Y_t(1 - H_i(t))h_i(H(t))dt \right].
\]

(A.2)

Further, we may rewrite the left hand side of (A.1) as

\[
\mathbb{E} \left[ \int_0^T Y_t dH_i(t) \right] = \mathbb{E}^P \left[ X_T \int_0^T Y_t dH_i(t) \right].
\]

(A.3)

Using Itô’s formula, it follows that for any \(0 \leq t \leq T\),

\[
d \left( X_t \int_0^t Y_s dH_i(s) \right) = X_{t-} Y_t dH_i(t) + \left( \int_0^{t-} Y_s dH_i(s) \right) dX_t + X_{t-} \left( \sum_{i=1}^M \lambda_i(H(t-)) \Delta H_i(t) \right) Y_t H_i(t)
\]

\[
= X_{t-} Y_t dH_i(t) + \left( \int_0^{t-} Y_s dH_i(s) \right) dX_t + X_{t-} Y_t \lambda_i(H(t-)) dH_i(t)
\]

\[
= X_{t-} Y_t (1 + \lambda_i(H(t-))) dH_i(t) + \left( \int_0^{t-} Y_s dH_i(s) \right) dX_t,
\]

where we have used the fact that the default indicator processes \(H_1(t), \ldots, H_M(t)\) do not jump simultaneously. Since \(X = (X_t; t \geq 0)\) is a \((\mathbb{F}, \mathcal{G}_t^{\mathbb{H}})\)-martingale, we have

\[
\mathbb{E}^P \left[ X_T \int_0^T Y_s dH_i(s) \right] = \mathbb{E}^P \left[ \int_0^T X_{t-} Y_t (1 + \lambda_i(H(t-))) dH_i(t) \right] = \mathbb{E}^P \left[ \int_0^T X_{t-} Y_t (1 - H_i(t)) h_i(H(t)) dt \right].
\]

The above expression, along with (A.2) and (A.3), results in the equality (A.1).

Proof of Lemma 2.2.
Proof. Using Feynman-Kac formula, we obtain that the functions $\Phi^{(1)}(t, z)$ and $\Phi^{(2)}(t, z)$ satisfy, respectively,

$$
\frac{\partial}{\partial t} + A \Phi^{(1)}_i(t, z) + (1 - z_i) = r \Phi^{(1)}_i(t, z), \quad \Phi_i^{(1)}(T_i, z) = 0, \quad \text{and}
$$

$$
\frac{\partial}{\partial t} + A \Phi^{(2)}_i(t, z) = r(1 - z_i) \Phi^{(2)}_i(t, z), \quad \Phi_i^{(2)}(T_i, z) = z_i,
$$

where the operator $A$ is defined by

$$
Ag(z) = \sum_{j=1}^{M}(1 - z_j)h_j(z) \left[ g(z^j) - g(z) \right], \quad z \in S.
$$

Then the function $\tilde{\Phi}^{L,\nu}_i(t, z)$ satisfies

$$
\frac{\partial}{\partial t} + A \tilde{\Phi}^{L,\nu}_i(t, z) = L_i \left( \frac{\partial}{\partial t} + A \right) \tilde{\Phi}^{L,\nu}_i(t, z) = rL_i(1 - z_i) \tilde{\Phi}^{L,\nu}_i(t, z) - r\nu_i \tilde{\Phi}^{L,\nu}_i(t, z) + \nu_i(1 - z_i)
$$

Using Itô’s formula, for all $t \geq 0$ we have

$$
\tilde{\Phi}^{L,\nu}_i(t, H(t)) = \tilde{\Phi}^{L,\nu}_i(0, H(0)) + \int_0^t \left( \frac{\partial}{\partial s} + A \right) \tilde{\Phi}^{L,\nu}_i(s, H(s))ds
$$

$$
+ \sum_{j=1}^{M} \int_0^t \left[ \tilde{\Phi}^{L,\nu}_i(s, H'(s)) - \tilde{\Phi}^{L,\nu}_i(s, H(s)) \right] d\xi_j(s)
$$

which corresponds with the equality in (11).

\[ \square \]

**Explicit solutions to Feynmann-Kac equations (A.4).**

**Lemma A.1.** Let $M \geq 2$. Then the Feynmann-Kac equations (A.4) admit the solutions given next. For any $i = 1, \ldots, M$,

- Let $z = 1$. Then it holds that

  $$
  \Phi_i^{(1)}(t, 1) = 0, \quad \text{and} \quad \Phi_i^{(2)}(t, 1) = 1.
  $$

- Let $z = 0^{j_1, \ldots, j_{M-1}}$, where $j_1, \ldots, j_{M-1} \in \{1, \ldots, M\}$ satisfy $j_1 \neq \cdots \neq j_{M-1}$. Set $j_M := \{1, \ldots, M\} \setminus \{j_1, \ldots, j_{M-1}\}$. Hence, if $i \neq j_M$,

  $$
  \Phi_i^{(1)}(t, 0^{j_1, \ldots, j_{M-1}}) = 0, \quad \text{and} \quad \Phi_i^{(2)}(t, 0^{j_1, \ldots, j_{M-1}}) = 1.
  $$

If $i = j_M$, then it holds that

$$
\Phi_i^{(1)}(t, 0^{j_1, \ldots, j_{M-1}}) = \frac{1}{p_{j_1, \ldots, j_{M-1}}} \left( 1 - e^{-p_{j_1, \ldots, j_{M-1}}(T_i - t)} \right),
$$

$$
\Phi_i^{(2)}(t, 0^{j_1, \ldots, j_{M-1}}) = \frac{h_{jm}(0^{j_1, \ldots, j_{M-1}})}{p_{j_1, \ldots, j_{M-1}}} \left( 1 - e^{-p_{j_1, \ldots, j_{M-1}}(T_i - t)} \right),
$$

\[ \square \]
Let $z = 0^{j_1, \ldots, j_m}$ with $0 \leq m \leq M - 2$ (where $0^{j_1, \ldots, j_m} = 0$ if $m = 0$). Set $\{j_{m+1}, \ldots, j_M\} := \{1, \ldots, M\} \setminus \{j_1, \ldots, j_m\}$. Hence, if $i \neq j_{m+1}, \ldots, j_M$, it holds that

$$
\Phi_i^{(1)}(t, 0^{j_1, \ldots, j_m}) = 0, \quad \text{and} \quad \Phi_i^{(2)}(t, 0^{j_1, \ldots, j_m}) = 1.
$$

If $i \in \{j_{m+1}, \ldots, j_M\}$, then it holds that

$$
\Phi_i^{(1)}(t, 0^{j_1, \ldots, j_m}) = e^{-p_{j_1, \ldots, j_m}(T_i - t)} \int_t^{T_i} Q_{i,j_1,\ldots,j_m}(s) e^{p_{j_1, \ldots, j_m}(T_i - s)} ds,
$$

$$
\Phi_i^{(2)}(t, 0^{j_1, \ldots, j_m}) = e^{-p_{j_1, \ldots, j_m}(T_i - t)} \int_t^{T_i} Q_{i,j_1,\ldots,j_m}(s) e^{p_{j_1, \ldots, j_m}(T_i - s)} ds,
$$

where the coefficients

$$
p_{j_1, \ldots, j_m} := r + \sum_{k=j_{m+1}, \ldots, j_M} h_k(0^{j_1, \ldots, j_m}),
$$

$$
Q_{i,j_1,\ldots,j_m}^{(1)}(t) := \sum_{k=j_{m+1}, \ldots, j_M, k \neq i} h_k(0^{j_1, \ldots, j_m}) \Phi_i^{(1)}(t, 0^{j_1, \ldots, j_m,k}) + 1,
$$

$$
Q_{i,j_1,\ldots,j_m}^{(2)}(t) := \sum_{k=j_{m+1}, \ldots, j_M, k \neq i} h_k(0^{j_1, \ldots, j_m}) \Phi_i^{(1)}(t, 0^{j_1, \ldots, j_m,k}) + h_i(0^{j_1, \ldots, j_m}).
$$

Proof. We consider the solution to the above equations iteratively.

- The case where $z = 1$: In this case, we have that for all $i \in \{1, \ldots, M\}$,

$$
\Phi_i^{(1)}(t, 1) = 0, \quad \text{and} \quad \Phi_i^{(2)}(t, 1) = 1.
$$

- The case where $z = 0^{j_1, \ldots, j_{M-1}}$: Here $j_1, \ldots, j_{M-1} \in \{1, \ldots, M\}$ satisfying $j_1 \neq \cdots \neq j_{M-1}$. Let $j_M := \{1, \ldots, M\} \setminus \{j_1, \ldots, j_{M-1}\}$. Hence, if $i \neq j_M$,

$$
\Phi_i^{(1)}(t, 0^{j_1, \ldots, j_{M-1}}) = 0, \quad \text{and} \quad \Phi_i^{(2)}(t, 0^{j_1, \ldots, j_{M-1}}) = 1.
$$

If $i = j_M$, then

$$
\Phi_i^{(1)}(t, 0^{j_1, \ldots, j_{M-1}}) = \frac{1}{p_{j_1,\ldots,j_{M-1}}} \left(1 - e^{-p_{j_1,\ldots,j_{M-1}}(T_i - t)}\right),
$$

$$
\Phi_i^{(2)}(t, 0^{j_1, \ldots, j_{M-1}}) = \frac{h_{j_M}(0^{j_1, \ldots, j_{M-1}})}{p_{j_1,\ldots,j_{M-1}}} \left(1 - e^{-p_{j_1,\ldots,j_{M-1}}(T_i - t)}\right),
$$

where $p_{j_1,\ldots,j_{M-1}} = r + h_{j_M}(0^{j_1, \ldots, j_{M-1}})$.

- The case where $z = 0^{j_1, \ldots, j_m}$ with $0 \leq m \leq M - 2$ (where $0^{j_1, \ldots, j_m} = 0$ if $m = 0$): Let $\{j_{m+1}, \ldots, j_M\} := \{1, \ldots, M\} \setminus \{j_1, \ldots, j_m\}$. Hence, if $i \neq j_{m+1}, \ldots, j_M$, it holds that

$$
\Phi_i^{(1)}(t, 0^{j_1, \ldots, j_m}) = 0, \quad \text{and} \quad \Phi_i^{(2)}(t, 0^{j_1, \ldots, j_m}) = 1.
$$

If $i \in \{j_{m+1}, \ldots, j_M\}$, then the function $\Phi_i^{(1)}(t, 0^{j_1, \ldots, j_m})$ should satisfy

$$
\frac{d}{dt} \Phi_i^{(1)}(t, 0^{j_1, \ldots, j_m}) = p_{j_1,\ldots,j_m} \Phi_i^{(1)}(t, 0^{j_1, \ldots, j_m}) - Q_{i,j_1,\ldots,j_m}^{(1)}(t),
$$

$$
\Phi_i^{(1)}(T_i, 0^{j_1, \ldots, j_m}) = 0,
$$

where

$$
p_{j_1,\ldots,j_m} := r + \sum_{k=j_{m+1}, \ldots, j_M} h_k(0^{j_1, \ldots, j_m}),
$$

$$
Q_{i,j_1,\ldots,j_m}^{(1)}(t) := \sum_{k=j_{m+1}, \ldots, j_M, k \neq i} h_k(0^{j_1, \ldots, j_m}) \Phi_i^{(1)}(t, 0^{j_1, \ldots, j_m,k}) + 1. \quad (A.6)
$$

Therefore, we have

$$
\Phi_i^{(1)}(t, 0^{j_1, \ldots, j_m}) = e^{-p_{j_1,\ldots,j_m}(T_i - t)} \int_t^{T_i} Q_{i,j_1,\ldots,j_m}^{(1)}(s) e^{p_{j_1,\ldots,j_m}(T_i - s)} ds.
$$
If \( i \in \{j_{m+1}, \ldots, j_M\} \), then the function \( \Phi^{(2)}_i(t, 0^{j_1 \ldots j_m}) \) should satisfy
\[
\frac{d}{dt} \Phi^{(2)}_i(t, 0^{j_1 \ldots j_m}) = p_{j_1 \ldots j_m} \Phi^{(2)}_i(t, 0^{j_1 \ldots j_m}) - Q^{(2)}_{i, j_1 \ldots j_m}(t),
\]
where
\[
Q^{(2)}_{i, j_1 \ldots j_m}(t) := \sum_{k=j_{m+1}, \ldots, j_M, k \neq i} h_k(0^{j_1 \ldots j_m}) \Phi^{(1)}_i(t, 0^{j_1 \ldots j_m, k}) + h_i(0^{j_1 \ldots j_m}). \quad (A.7)
\]
Therefore, we have
\[
\Phi^{(2)}_i(t, 0^{j_1 \ldots j_m}) = e^{-p_{j_1 \ldots j_m}(T_t - t)} \int_t^{T_t} Q^{(2)}_{i, j_1 \ldots j_m}(s) e^{p_{j_1 \ldots j_m}(T_t - s)} ds.
\]
Thus we complete the proof of the lemma. \( \square \)

**Corollary A.2.** Let \( M = 2 \). Set \( p_1 = r + h_1(0, 0) + h_2(0, 0) \), \( p_2 = r + h_2(1, 0) \), \( p_3 = r + h_1(0, 1) \). Then, the Feynman-Kac equations (A.4) admit an explicit representation
\[
\begin{align*}
\Phi^{(1)}_1(t, 1, 0) &= 0, \quad \Phi^{(2)}_1(t, 1, 0) = 1, \quad \text{and hence } \Phi^{L,v}_1(t, 1, 0) := L_1 \Phi^{(2)}_1(t, 1, 0) - \nu_1 \Phi^{(1)}_1(t, 1, 0) = L_1, \\
\Phi^{(1)}_1(t, 1, 1) &= 0, \quad \Phi^{(2)}_1(t, 1, 1) = 1, \quad \text{and hence } \Phi^{L,v}_1(t, 1, 1) := L_1 \Phi^{(2)}_1(t, 1, 1) - \nu_1 \Phi^{(1)}_1(t, 1, 1) = L_1, \\
\Phi^{(1)}_1(t, 0, 1) &= \frac{1}{p_3} \left(1 - e^{-p_3(T_t - t)}\right), \quad \Phi^{(2)}_1(t, 0, 1) = \frac{h_1(0, 1)}{p_3} \left(1 - e^{-p_3(T_t - t)}\right),
\end{align*}
\]
and hence
\[
\Phi^{L,v}_1(t, 0, 1) := L_1 \Phi^{(2)}_1(t, 0, 1) - \nu_1 \Phi^{(1)}_1(t, 0, 1) = \frac{L_1 h_1(0, 1) - \nu_1}{p_3} \left(1 - e^{-p_3(T_t - t)}\right). \quad (A.8)
\]

- **The case** \( p_1 \neq p_3 \),
  \[
  \begin{align*}
  \Phi^{(1)}_1(t, 0, 0) &= E e^{-p_3(T_t - t)} - (E + F)e^{-p_1(T_t - t)} + F, \\
  \Phi^{(2)}_1(t, 0, 0) &= G e^{-p_3(T_t - t)} - (G + H)e^{-p_1(T_t - t)} + H, \\
  \Phi^{L,v}_1(t, 0, 0) &= (L_1 G - \nu_1 E) e^{-p_3(T_t - t)} - (L_1 G - \nu_1 E + L_1 H - \nu_1 F) e^{-p_1(T_t - t)} + (L_1 H - \nu_1 F),
  \end{align*}
  \]
  where
  \[
  E = \frac{h_2(0, 0)}{p_3(p_3 - p_1)}, \quad F = \frac{h_2(0, 0) + p_3}{p_3 p_1}, \quad G = \frac{h_1(0, 1) h_2(0, 0)}{p_3(p_3 - p_1)}, \quad H = \frac{h_1(0, 1) h_2(0, 0) + h_1(0, 0) p_3}{p_1 p_3},
  \]

- **The case** \( p_1 = p_3 \),
  \[
  \begin{align*}
  \Phi^{(1)}_1(t, 0, 0) &= F \left(1 - e^{-p_3(T - t)}\right) - \frac{h_2(0, 0)}{p_1} (T_t - t) e^{-p_1(T_t - t)}, \\
  \Phi^{(2)}_1(t, 0, 0) &= H \left(1 - e^{-p_3(T - t)}\right) - \frac{h_1(0, 1) h_2(0, 0)}{p_1} (T_t - t) e^{-p_1(T_t - t)}, \\
  \Phi^{L,v}_1(t, 0, 0) &= (L_1 H - \nu_1 F) \left(1 - e^{-p_1(T - t)}\right) - \frac{(L_1 h_2(0, 0) - \nu_1) h_2(0, 0)}{p_1} (T_t - t) e^{-p_1(T_t - t)},
  \end{align*}
  \]
  and
  \[
  \begin{align*}
  \Phi^{(1)}_2(t, 0, 1) &= 0, \quad \Phi^{(2)}_2(t, 0, 1) = 1, \quad \text{and hence } \Phi^{L,v}_2(t, 0, 1) = L_2, \\
  \Phi^{(1)}_2(t, 1, 1) &= 0, \quad \Phi^{(2)}_2(t, 1, 1) = 1, \quad \text{and hence } \Phi^{L,v}_2(t, 1, 1) = L_2, \\
  \Phi^{(1)}_2(t, 1, 0) &= \frac{1}{p_2} \left(1 - e^{-p_2(T - t)}\right), \quad \Phi^{(2)}_2(t, 1, 0) = \frac{h_2(1, 0)}{p_2} \left(1 - e^{-p_2(T - t)}\right),
  \end{align*}
  \]
  and hence
  \[
  \Phi^{L,v}_2(t, 1, 0) = \frac{L_2 h_2(1, 0) - \nu_2}{p_2} \left(1 - e^{-p_2(T - t)}\right). \quad (A.11)
  \]
• The case $p_1 \neq p_2$,
\[
\Phi_2^{(1)}(t, 0, 0) = Ce^{-p_2(T_2-t)} - (C + D)e^{-p_1(T_2-t)} + D,
\]
\[
\Phi_2^{(2)}(t, 0, 0) = Ae^{-p_2(T_2-t)} - (A + B)e^{-p_1(T_2-t)} + B,
\]
\[
\hat{\Phi}^{L, \nu}_i(t, 0, 0) = (L_2A - \nu_2C)e^{-p_2(T_2-t)} - (L_2A - \nu_2C + L_2B - \nu_2D)e^{-p_1(T_2-t)} + (L_2B - \nu_2D),
\]
(A.12)

where
\[
C = \frac{h_1(0, 0)}{p_2(p_2 - p_1)}, \quad D = \frac{h_1(0, 0) + p_2}{p_1p_2},
\]
\[
A = \frac{h_1(0, 0)h_2(1, 0)}{p_2(p_2 - p_1)}, \quad B = \frac{h_1(0, 0)h_2(1, 0) + h_2(0, 0)p_2}{p_1p_2}.
\]

• The case $p_1 = p_2$,
\[
\Phi_2^{(1)}(t, 0, 0) = D \left(1 - e^{-p_1(T_2-t)}\right) - \frac{h_1(0, 0)}{p_1} (T_2 - t)e^{-p_1(T_2-t)},
\]
\[
\Phi_2^{(2)}(t, 0, 0) = B \left(1 - e^{-p_1(T_2-t)}\right) - \frac{h_2(1, 0)h_1(0, 0)}{p_1} (T_2 - t)e^{-p_2(T_1-t)},
\]
\[
\Phi_2^{(L, \nu)}(t, 0, 0) = (L_2B - \nu_2D) \left(1 - e^{-p_1(T_2-t)}\right) - \frac{(L_2h_2(1, 0) - \nu_2)h_1(0, 0)}{p_1} (T_2 - t)e^{-p_1(T_2-t)}.
\]
(A.13)

**Proof of Proposition 2.3.**

Proof. In light of the representation given in (7) and using Itô’s formula, we obtain
\[
dCDS_i^{(j)} = (1 - H_i(t-)) \sum_{j=1}^M \left[ \hat{\Phi}^{L, \nu}_i(t, H_i(t)) - \hat{\Phi}^{L, \nu}_i(t, H_i(t^-)) \right] dH_i(t) + \Delta H_i(t) \left[ \hat{\Phi}^{L, \nu}_i(t, H_i(t)) - \hat{\Phi}^{L, \nu}_i(t, H_i(t^-)) \right] dH_i(t),
\]
(A.14)

where we used the equality
\[
\Delta \hat{\Phi}^{L, \nu}_i(t, H_i(t)) dH_i(t) = \left[ \hat{\Phi}^{L, \nu}_i(t, H_i(t^-)) - \hat{\Phi}^{L, \nu}_i(t, H_i(t^-)) \right] dH_i(t),
\]

which follows from the fact that our default model excludes the occurrence of simultaneous defaults. Hence, if the $i$-th name defaults, any other name $j \neq i$ would not, and consequently $\Delta \hat{\Phi}^{L, \nu}_i(t, H_i(t)) = \hat{\Phi}^{L, \nu}_i(t, H_i(t^-)) - \hat{\Phi}^{L, \nu}_i(t, H_i(t^-))$. If $j \neq i$-name defaults, then $dH_i(t) = 0$ and hence the above equality still holds.

Using Eq. (11) in Lemma 2.2, we obtain
\[
(1 - H_i(t-)) \sum_{j=1}^M \left[ \hat{\Phi}^{L, \nu}_i(t, H_i(t^-)) - \hat{\Phi}^{L, \nu}_i(t, H_i(t^-)) \right] dH_i(t)
\]
\[
= (1 - H_i(t)) \left[ r(1 - H_i(t)) \hat{\Phi}^{L, \nu}_i(t, H_i(t)) - r\nu_i H_i(t) \Phi^{(1)}_i(t, H_i(t)) + \nu_i(1 - H_i(t)) \right] dt
\]
\[
+ (1 - H_i(t-)) \sum_{j=1}^M \left[ \hat{\Phi}^{L, \nu}_i(t, H_i(t^-)) - \hat{\Phi}^{L, \nu}_i(t, H_i(t^-)) \right] d\xi_j(t)
\]
\[
= \left[ r(1 - H_i(t)) \hat{\Phi}^{L, \nu}_i(t, H_i(t)) + \nu_i(1 - H_i(t)) \right] dt
\]
\[
+ (1 - H_i(t-)) \sum_{j=1}^M \left[ \hat{\Phi}^{L, \nu}_i(t, H_i(t^-)) - \hat{\Phi}^{L, \nu}_i(t, H_i(t^-)) \right] d\xi_j(t).
\]

From (A.14), we have
\[
dCDS_i^{(j)} = (1 - H_i(t-)) \sum_{j=1}^M \left[ \hat{\Phi}^{L, \nu}_i(t, H_i(t^-)) - \hat{\Phi}^{L, \nu}_i(t, H_i(t^-)) \right] dH_i(t)
\]
Using the equality $\Phi^{L^\nu}(t, z^1) = L_i$ following from Lemma A.1, we obtain the desired result.

Proof of Lemma 4.2

Proof. Denote $\{j_{m+1}, \ldots, j_M\}$ all elements of the set $\{1, \ldots, M\} \setminus \{j_1, \ldots, j_m\}$. We want to find the optimum $\tilde{\phi}^{CDS,*}_{i,j_1,\ldots,j_m}$ satisfying the conditions: for any $i = j_{m+1}, \ldots, j_M$,

$$\tilde{\phi}^{CDS,*}_{i,j_{m+1},\ldots,j_m}(t) > - \frac{1}{L_i - \Phi^{L^\nu}_{i}(t, 0^{j_{m+1}})} := M_1^{(i)}(t),$$

and

$$\tilde{\phi}^{CDS,*}_{i,j_{m+1},\ldots,j_m}(t) \left( \Phi^{L^\nu}_{i}(t, 0^{j_{m+1}}) - \Phi^{L^\nu}_{i}(t, 0^{j_{m+1}, \ell}) \right) > -1, \quad \forall \ell \in N_{i,j_{m+1},\ldots,j_m}. \quad (A.16)$$

We can assume that there exist only $\ell_1, \ldots, \ell_k \in N_{i,j_{m+1},\ldots,j_m}$ for some $k \in \{0, \ldots, |N_{i,j_{m+1},\ldots,j_m}|\}$ so that $\Phi^{L^\nu}_{i}(t, 0^{j_{m+1}, \ell_n}) < \Phi^{L^\nu}_{i}(t, 0^{j_{m+1}, \ldots, j_n})$ for all $n = 1, \ldots, k$. Then it holds that $\Phi^{L^\nu}_{i}(t, 0^{j_{m+1}, \ldots, j_n}) > \Phi^{L^\nu}_{i}(t, 0^{j_{m+1}, \ldots, j_n})$ for all $n = k+1, \ldots, |N_{i,j_{m+1},\ldots,j_m}|$.

Here if $k = 0$, then it implies that $\Phi^{L^\nu}_{i}(t, 0^{j_{m+1}, \ldots, j_n}) > \Phi^{L^\nu}_{i}(t, 0^{j_{m+1}, \ldots, j_n})$ for all $n = 1, \ldots, |N_{i,j_{m+1},\ldots,j_m}|$. If $k = |N_{i,j_{m+1},\ldots,j_m}|$, then it implies that $\Phi^{L^\nu}_{i}(t, 0^{j_{m+1}, \ldots, j_n}) < \Phi^{L^\nu}_{i}(t, 0^{j_{m+1}, \ldots, j_n})$ for all $n = 1, \ldots, |N_{i,j_{m+1},\ldots,j_m}|$.

Note that

$$\max \left\{ M_1^{(i)}(t), \frac{1}{\Phi^{L^\nu}_{i}(t, 0^{j_{m+1}})} - \Phi^{L^\nu}_{i}(t, 0^{j_{m+1}, \ldots, j_n}) ; \quad n = k+1, \ldots, |N_{i,j_{m+1},\ldots,j_m}| \right\} = M_1^{(i)}(t),$$

since $\Phi^{L^\nu}_{i}(t, 0^{j_{m+1}, \ldots, j_n}) < L_i$. Define

$$M_2^{(i,m)}(t) := \min \left\{ \frac{1}{\Phi^{L^\nu}_{i}(t, 0^{j_{m+1}})} - \Phi^{L^\nu}_{i}(t, 0^{j_{m+1}, \ldots, j_n}) ; \quad n = 1, \ldots, k \right\}, \quad (A.17)$$

where $M_2^{(i,m)}(t) = +\infty$ if the set $\{\}$ on the r.h.s. of the above assignment is empty (i.e. if $k = 0$). Hence for each $i \neq j_1, \ldots, j_m$, the optimum must satisfy

$$M_1^{(i)}(t) < \tilde{\phi}^{CDS,*}_{i,j_{m+1},\ldots,j_m}(t) < M_2^{(i,m)}(t). \quad (A.18)$$

Next we consider the existence and uniqueness of the optimum $\tilde{\phi}^{CDS,*}_{i,j_{m+1},\ldots,j_m}$ as a solution to the equation with unknown $\tilde{\phi}^{CDS}_{i,j_{m+1},\ldots,j_m}$ satisfying (37):

$$g_{i,j_{m+1},\ldots,j_m} \left( \tilde{\phi}^{CDS}_{i,j_{m+1},\ldots,j_m}, t, B^{(j_{m+1},\ldots,j_m,i)} \right) = 0. \quad (A.19)$$

From the expression of the function $g_{i,j_{m+1},\ldots,j_m}$ defined by (36), we see that it is decreasing w.r.t. $\tilde{\phi}^{CDS}_{i,j_{m+1},\ldots,j_m}$. We have the following sub-cases:

- If $M_2^{(i,m)}(t) < +\infty$, then we have

$$\lim_{\phi^{CDS}_{i,j_{m+1},\ldots,j_m} \to M_1^{(i)}(t)} g_{i,j_{m+1},\ldots,j_m} \left( \phi^{CDS}_{i,j_{m+1},\ldots,j_m}, t, B^{(j_{m+1},\ldots,j_m,i)} \right) = +\infty,$$

and

$$\lim_{\phi^{CDS}_{i,j_{m+1},\ldots,j_m} \to M_2^{(i,m)}(t)} g_{i,j_{m+1},\ldots,j_m} \left( \phi^{CDS}_{i,j_{m+1},\ldots,j_m}, t, B^{(j_{m+1},\ldots,j_m,i)} \right) = -\infty.$$

Since $g_{i,j_{m+1},\ldots,j_m} \left( \tilde{\phi}^{CDS}_{i,j_{m+1},\ldots,j_m}, t, B^{(j_{m+1},\ldots,j_m,i)} \right)$ is continuous in $\tilde{\phi}^{CDS}_{i,j_{m+1},\ldots,j_m} \in (M_1^{(i)}(t), M_2^{(i,m)}(t))$, we can apply the Intermediate Value Theorem, and obtain a unique solution $\tilde{\phi}^{CDS}_{i,j_{m+1},\ldots,j_m} \in (M_1^{(i)}(t), M_2^{(i,m)}(t))$ to the equation (A.19).
If $M_2^{(t,m)}(t) = +\infty$, then it holds that $\hat{\Phi}^{L,B}(t,0^{1\ldots,j_m}) > \hat{\Phi}^{L,B}(t,0^{1\ldots,j_m})$ for all $t \in N_{i,j_1\ldots,j_m}$.

Hence
\[
\lim_{\phi^{CD}_B(t) \rightarrow 1,m}(t) g_{i,j_1\ldots,j_m}(\hat{\phi}^{CD}_B(t),\phi^{CD}_B(t),t,B^{(j_1\ldots,j_m)}) = +\infty,
\]
and
\[
\lim_{\phi^{CD}_B(t) \rightarrow 1,m} g_{i,j_1\ldots,j_m} = V_{i,j_1\ldots,j_m}(t)B^{(j_1\ldots,j_m)}.
\]

Since it holds that $\hat{\Phi}^{L,B}(t,0^{1\ldots,j_m}) > \hat{\Phi}^{L,B}(t,0^{1\ldots,j_m})$ for all $t \in N_{i,j_1\ldots,j_m}$, we have
\[
V_{i,j_1\ldots,j_m}(t) := \left(\hat{\Phi}^{L,B}(t,0^{1\ldots,j_m}) - L_i\right) h_{i}(0^{1\ldots,j_m}) - \sum_{t \in N_{i,j_1\ldots,j_m}} \left(\hat{\Phi}^{L,B}(t,0^{1\ldots,j_m}) - \hat{\Phi}^{L,B}(t,0^{1\ldots,j_m})\right) h_{t}(0^{1\ldots,j_m}) < 0,
\]
Thus $V_{i,j_1\ldots,j_m}(t)B^{(j_1\ldots,j_m)} < 0$, since $B^{(j_1\ldots,j_m)} > 0$.

Hence, we have that there exists a unique solution $\hat{\phi}^{CD}_B(t) \rightarrow 1,m$ to Eq. (A.19) in the desired domain of $\phi^{CD}_B(t) \rightarrow 1,m$. In light of Kumagai (1980)’s implicit function theorem, we will also have that $\phi^{CD}_B(t) \rightarrow 1,m$ is continuous if we prove that $g_{i,j_1\ldots,j_m}(\hat{\phi}^{CD}_B(t),\phi^{CD}_B(t),t,B^{(j_1\ldots,j_m)})$ is continuous in $\phi^{CD}_B(t) \rightarrow 1,m$, given a suitably chosen constant $\delta_1 > 0$, and the function $x^\gamma y$ is continuous in $x$. Since composition of continuous functions is continuous, we obtain that the desired result.

**Proof of Lemma 4.3.**

**Proof.** First define
\[
G(t, u) := K_{jm}(\gamma)u + A_{jm}(t, \gamma)u^\gamma + U_{jm}(t, \gamma), \quad (t, u) \in \mathbb{R} \times \mathbb{R}_+,
\]
where we extend the coefficients $A_{jm}(t, \gamma)$ and $U_{jm}(t, \gamma)$, defined on $t \in [0, T]$ to $\mathbb{R}$ by setting $A_{jm}(t, \gamma) = A_{jm}(0, \gamma)$, $U_{jm}(t, \gamma) = U_{jm}(0, \gamma)$ for $t < 0$ and $A_{jm}(t, \gamma) = A_{jm}(T, \gamma)$, $U_{jm}(t, \gamma) = U_{jm}(T, \gamma)$ for $t > T$.

Then the functions $G(t, u)$ and $\frac{\partial G(t, u)}{\partial t}$ are continuous on the domain $(t, u) \in \mathcal{O} := \mathbb{R} \times \mathbb{R}_+$ containing the point $(T, \gamma_{+1})$. By virtue of the fundamental theorem of existence and uniqueness of the solution (see, e.g. Chicone (2006)) to the following equation
\[
u'(t) = G(t, \nu(t)), \quad \nu(T) = \gamma_{+1} > 0, \quad (A.20)
\]
we obtain that there exists a unique solution to Eq. (A.20) for $t \in [T - \delta_1, T + \delta_1]$, given a suitably chosen constant $\delta_1 > 0$. Let $y(l) = \hat{u}(l)e^{-K_{jm}(\gamma)t}$. Then we have
\[
y'(l) = e^{-K_{jm}(\gamma)t} \left(A_{jm}(t, \gamma)e^{\gamma K_{jm}(\gamma)t}y(t) + U_{jm}(t, \gamma)\right), \quad y(T) = \gamma_{+1}e^{-K_{jm}(\gamma)T} > 0.
\]
Note that $A_{jm}(t, \gamma) < 0$ and $U_{jm}(t, \gamma) < 0$ for all $t \in \mathbb{R}$. Hence $y'(T) < 0$, since $y(T) > 0$. Using the continuity of the solution $\nu'(t)$ at $T$, there exists $\delta_1 > 0$ (possibly smaller than $\delta_1$) so that $y'(t) < 0$ for all $t \in [T - \delta_2, T + \delta_2]$. Since $y(T) > 0$, we have that $y(t) > 0$ for all $t \in [T - \delta_1, T]$. For the case of $\delta_1 \leq \delta_1$, we can repeat the above argument and show that $y(t) > 0$ for all $t \in [T - \delta_1, T]$. This implies that $\nu(t) > 0$ for all $t \in [T - \delta_1, T]$.

Based on the above proven existence and uniqueness of a positive local solution $\nu(t)$ for $t \in [T - \delta_1, T + \delta_1]$, we now show existence and uniqueness of a global solution on $[0, T]$. This is done solving repeatedly solving the equation $\nu(t) = G(t, \nu(t))$ backward. If $\delta_1 \geq T$, then we have obtained the unique positive $\nu(t)$ for $t \in [0, T] \subset [T - \delta_1, T + \delta_1]$. If $\delta_1 < T$, we consider $v'(t) = G(t, v(t))$ on $t \in [0, T - \delta_1]$ with terminal condition $v(T - \delta_1) = \nu(T - \delta_1) > 0$. Since $G(t, v)$ and $\frac{\partial G(t, v)}{\partial t}$ are continuous on the domain $(t, v) \in \mathcal{O}$ containing the point $(T - \delta_1, \nu(T - \delta_1))$, we can repeat an argument similar to the one above, and obtain a unique positive solution on $[T - \delta_1 - \delta_2, T - \delta_1]$ for some positive $\delta_2 > 0$. Hence, we have the solution on $[T - \delta_1 - \delta_2, T - \delta_1] \cup [T - \delta_1, T] = [T - \delta_1 - \delta_2, T]$. Iterating such procedure, we obtain the solution on the whole interval $[0, T]$. \qed
Proof of Lemma 4.5

Proof. Using that \( \tilde{\Phi}_i^{L,V}(t, 0^{j_1, \ldots, j_M}) = L_i \), define for \( i \in \{ j_{m+1}, \ldots, j_M \} := \{1, \ldots, M\} \setminus \{j_1, \ldots, j_m\} \),

\[
C_{j_1, \ldots, j_m}(t, u) := \begin{cases}
\gamma \sum_{i=j_{m+1}, \ldots, j_M} \tilde{\phi}_{CDS,\ast}^{i}(t, u, \tilde{B}^{(j_1, \ldots, j_{m+1})}(t)) \left( \left( \tilde{\Phi}_i^{L,V}(t, 0^{j_1, \ldots, j_{m+1}}) - L_i \right) h_i(0^{j_1, \ldots, j_m}) \right)
- \sum_{\ell \in N_{j_1, \ldots, j_m}} \left( \sum_{i=j_{m+1}, \ldots, j_M} h_i^\ell (0^{j_1, \ldots, j_{m+1}}) \right) h_i(0^{j_1, \ldots, j_m}) + a(\gamma)
- \sum_{i=j_{m+1}, \ldots, j_M} \sum_{t_i \in J_{j_1, \ldots, j_m}} \left( \sum_{i=j_{m+1}, \ldots, j_M} h_i^\ell (0^{j_1, \ldots, j_{m+1}}) \right) u
+ \sum_{i=j_{m+1}, \ldots, j_M} \left( 1 + \phi_{CDS,\ast}^{i,j_1, \ldots, j_m} \left( t, u, \tilde{B}^{(j_1, \ldots, j_{m+1})}(t) \right) \left( L_i - \tilde{\Phi}_i^{L,V}(t, 0^{j_1, \ldots, j_{m+1}}) \right) \right) \tilde{B}_i^{j_1, \ldots, j_{m+1}}(t) h_i^\ell (0^{j_1, \ldots, j_{m+1}})
+ \sum_{i=j_{m+1}, \ldots, j_M} \left( \sum_{t_i \in J_{j_1, \ldots, j_m}} \left( 1 + \phi_{CDS,\ast}^{i,j_1, \ldots, j_m} \left( t, u, \tilde{B}^{(j_1, \ldots, j_{m+1})}(t) \right) \left( \tilde{\Phi}_i^{L,V}(t, 0^{j_1, \ldots, j_{m+1}}) - \tilde{\Phi}_i^{L,V}(t, 0^{j_1, \ldots, j_{m+1}}) \right) \right) \right) \tilde{B}_i^{j_1, \ldots, j_{m+1}}(t) h_i^\ell (0^{j_1, \ldots, j_{m+1}})
\end{cases}
\]

where we have used the notation:

\[
\tilde{B}^{(j_1, \ldots, j_{m+1})}(t) := \left[ B^{(0^{j_1, \ldots, j_{m+1}})}(t), B^{(0^{j_1, \ldots, j_{m+1}})}(t); \ell \in N_{j_1, \ldots, j_{m+1}} \right]^T.
\]

We have assumed that for each tuple \( (j_1, \ldots, j_{m+1}) \) satisfying \( j_1 \neq \cdots \neq j_{m+1} \), there exists a unique positive \( B^{(0^{j_1, \ldots, j_{m+1}})}(t) \) to Eq. (42). Hence \( B^{(0^{j_1, \ldots, j_{m+1}})}(t) \) has continuous first-order partial derivatives with respect to \( t \in [0, T] \). It remains to prove that the optimal strategies

\[
\tilde{\phi}_{CDS,\ast}^{i,j_1, \ldots, j_m}(t, u) := \tilde{\phi}_{CDS,\ast}^{i,j_1, \ldots, j_m} \left( t, u, \tilde{B}^{(j_1, \ldots, j_{m+1})}(t) \right), \quad i = j_{m+1}, \ldots, j_M
\]

have continuous first-order partial derivatives with respect to \( (t, u) \in [0, T] \times \mathbb{R}_+ \). By virtue of Lemma 4.2, we know that the optimum \( \phi_{CDS,\ast}^{i,j_1, \ldots, j_m}(t, \tilde{B}^{(j_1, \ldots, j_{m+1})}) \) is continuous with respect to \( (t, \tilde{B}^{(j_1, \ldots, j_{m+1})}) \) and hence \( \phi_{CDS,\ast}^{i,j_1, \ldots, j_m}(t, u) \) is continuous with respect to \( (t, u) \in [0, T] \times \mathbb{R}_+ \). Recall \( g_{j_1, \ldots, j_m} \left( \phi_{CDS}^{i,j_1, \ldots, j_m}, t, u, \tilde{B}^{(j_1, \ldots, j_{m+1})}(t) \right) \) defined by (36). Then \( g_{j_1, \ldots, j_m} \left( \phi_{CDS}^{i,j_1, \ldots, j_m}, t, u, \tilde{B}^{(j_1, \ldots, j_{m+1})}(t) \right) \) are continuously differentiable on the set \( (\phi_{CDS}^{i,j_1, \ldots, j_m}, t, u) \in D^1 \times [0, T] \times \mathbb{R}_+ \), where \( D^1 \) is the admissibility domain of the optimum \( \phi_{CDS,\ast}^{i,j_1, \ldots, j_m} \), specified by (37). Using Lemma 4.2 and the Implicit Function Theorem, it follows that the optimum \( \phi_{CDS,\ast}^{i,j_1, \ldots, j_m}(t, u) \) defined by (A.22) is continuously differentiable on \([0, T] \times \mathbb{R}_+ \). By virtue of the fundamental theorem of existence and uniqueness of the local solution to the following equation:

\[
u'(t) = -C_{j_1, \ldots, j_m}(t, u(t)), \quad u(T) = \frac{1}{\gamma},
\]

we have that there exists a unique solution to Eq. (A.23) for \( t \in [T - \delta_1, T + \delta_1] \) for some constant \( \delta_1 > 0 \). Note that we can write the function \( C_{j_1, \ldots, j_m}(t, u(t)) \) in the following form:

\[
C_{j_1, \ldots, j_m}(t, u) = f_{j_1, \ldots, j_m}(t, u) + g_{j_1, \ldots, j_m}(t, u) + v_{j_1, \ldots, j_m}(t),
\]

where the coefficients

\[
f_{j_1, \ldots, j_m}(t, u) := \gamma \sum_{i=j_{m+1}, \ldots, j_M} \tilde{\phi}_{CDS,\ast}^{i,j_1, \ldots, j_m} \left( t, u, \tilde{B}^{(j_1, \ldots, j_{m+1})}(t) \right) \left( \tilde{\Phi}_i^{L,V}(t, 0^{j_1, \ldots, j_{m+1}}) - L_i \right) h_i(0^{j_1, \ldots, j_m})
- \sum_{\ell \in N_{j_1, \ldots, j_m}} \left( \sum_{i=j_{m+1}, \ldots, j_M} h_i^\ell (0^{j_1, \ldots, j_{m+1}}) \right) h_i(0^{j_1, \ldots, j_m}) + a(\gamma)
\]

\[
g_{j_1, \ldots, j_m}(t, u) := \sum_{i=j_{m+1}, \ldots, j_M} \tilde{\phi}_{CDS,\ast}^{i,j_1, \ldots, j_m} \left( t, u, \tilde{B}^{(j_1, \ldots, j_{m+1})}(t) \right) \left( \tilde{\Phi}_i^{L,V}(t, 0^{j_1, \ldots, j_{m+1}}) - \tilde{\Phi}_i^{L,V}(t, 0^{j_1, \ldots, j_{m+1}}) \right) h_i(0^{j_1, \ldots, j_m})
\]

\[
v_{j_1, \ldots, j_m}(t) := \sum_{i=j_{m+1}, \ldots, j_M} \tilde{\phi}_{CDS,\ast}^{i,j_1, \ldots, j_m} \left( t, u, \tilde{B}^{(j_1, \ldots, j_{m+1})}(t) \right) \left( \tilde{\Phi}_i^{L,V}(t, 0^{j_1, \ldots, j_{m+1}}) - \tilde{\Phi}_i^{L,V}(t, 0^{j_1, \ldots, j_{m+1}}) \right) h_i(0^{j_1, \ldots, j_m})
\]
Recall that we have assumed that $A_{27}$

\[
q_{j_1,...,j_m}(t,u) := \sum_{i=m+1,...,J_M} \left[ 1 + \phi_i^{CDS,\ast}(t,u,\hat{u}_{j_1,...,j_m}(t))(L_i - \hat{v}_i^o(t,0^{j_1,...,j_m})) \right]^\gamma 
\times \hat{v}_{i}^{(j_1,...,j_m)}(t)h_i^o(0^{j_1,...,j_m}) 
+ \sum_{i=m+1,...,J_M} \left[ \sum_{\ell \in N_{i,j_1,...,j_m}} \left[ 1 + \phi_i^{CDS,\ast}(t,u,\hat{u}_{j_1,...,j_m}(t))(L_i - \hat{v}_i^o(t,0^{j_1,...,j_m})) \right]^\gamma \hat{v}_{i}^{(j_1,...,j_m)}(t)h_i^o(0^{j_1,...,j_m}) \right],
\]

for all $t \in [T - \delta_1, T + \delta_1]$. Then we have that for $t \in [T - \delta_1, T + \delta_1]$, it holds that

\[
u_{j_1,...,j_m}(t) := \sum_{i=m+1,...,J_M} \left[ \sum_{\ell \in N_{i,j_1,...,j_m}} \hat{v}_{i}^{(j_1,...,j_m)}(t)h_i^o(0^{j_1,...,j_m}) \right].
\]

(A.25)

Define the function $\tilde{u}(t) = u(t) - \int_{t}^{T} f_{j_1,...,j_m}(s,u(s))ds$, where $u(t)$ is the solution to Eq. (A.23) in time $t \in [T - \delta_1, T + \delta_1]$. Then we have that for $t \in [T - \delta_1, T + \delta_1]$, it holds that

\[
u'(T) = \frac{1}{\gamma} > 0.
\]

(A.26)

Recall that we have assumed that $\hat{v}_{i}^{(j_1,...,j_m)}(t)$ is positive for all $\ell = 1, j_{m+1}, \ldots, J_M$ and $i = j_1, \ldots, j_m$. This leads to $\nu_{j_1,...,j_m}(t) > 0$. Moreover, since the optimum $\hat{v}_{i}^{CDS,\ast}(t,u,\hat{u}_{j_1,...,j_m}(t))(L_i - \hat{v}_i^o(t,0^{j_1,...,j_m})) > 0$ and $1 + \phi_i^{CDS,\ast}(t,u,\hat{u}_{j_1,...,j_m}(t))(L_i - \hat{v}_i^o(t,0^{j_1,...,j_m})) > 0$ for each $i = j_{m+1}, \ldots, J_M$, using (37), we also have $q_{j_1,...,j_m}(t,u(t)) > 0$. This results in $\nu'(T) < 0$. Using the continuity of the function $u(t)$ at time $t = T$, we have that there exists a $\delta_1 > 0$ (possibly smaller than $\delta_1$) so that $\nu'(T) < 0$ for all $t \in [T - \delta_1, T]$. Thus we have $\tilde{u}(t) > 0$ for all $t \in [T - \delta_1, T]$ and hence $u(t) = \tilde{u}(t)e^{-\int_t^T f_{j_1,...,j_m}(s,u(s))ds} > 0$ for all $t \in [T - \delta_1, T]$. If $\delta_1 \geq \delta_1$, then we indeed have $u(t) > 0$ for all $t \in [T - \delta_1, T]$. For the case of $\delta_1 < \delta_1$, we can repeat the above argument to show the positivity of $u(t)$ when $t \in [T - \delta_1, T - \delta_1]$ and thus we still have $u(t) > 0$ for all $t \in [T - \delta_1, T]$.

Based on the existence and uniqueness of a positive local solution $u(t)$ to Eq. (A.23) proven above, we next show existence and uniqueness of a global solution $[0,T]$ by repeatedly solving Eq. (A.23) backward. If $\delta_1 \geq T$, then we have obtained the unique positive $u(t)$ for $t \in [0,T] \subset [T - \delta_1, T]$. If $\delta_1 < T$, we consider the Eq. $v(t) = -C_{j_1,...,j_m}(t,v(t))$ on $t \in [0,T - \delta_1]$ with the terminal condition $v(T - \delta_1) = u(T - \delta_1) > 0$. Since $C_{j_1,...,j_m}(t,u)$ and $\frac{\partial C_{j_1,...,j_m}(t,u)}{\partial u}$ are continuous on the domain $(t,u) \in [0,T] \times \mathbb{R}_+$ containing the point $(T - \delta_1, u(T - \delta_1))$, we can repeat an argument similar to above, and obtain a unique positive solution on $t \in [T - \delta_1, T - \delta_1]$ for some constant $\delta_2 > 0$. Hence, we thus have the positive solution on $t \in [T - \delta_1, T - \delta_2] \cup [T - \delta_1, T] = [T - \delta_1, T - \delta_2, T]$. Iterating such procedure, we obtain the solution on the whole interval $[0,T]$.

Recalling that $B(0^{j_1,...,j_m}) = \hat{u}(t) > 0$ for all $t \in [0,T]$, we deduce a unique positive solution $B(0^{j_1,...,j_m})$ to Eq. (42).

Proof of Proposition 1.3

We first solve Eq. (61). Recall that $a(\gamma) = \gamma + \frac{(r - \mu)^2}{2(1 - \gamma)^2} > 0$. Then (61) may be reduced to

\[
\frac{\partial B(1^{1,1})(t)}{\partial t} = -a(\gamma) B(1^{1,1})(t), \quad B(1^{1,1})(T) = 1, \gamma.
\]

Obviously, we have that $B(1^{1,1})(t) = 1 + e^{a(\gamma)(T - t)}$ for $0 \leq t \leq T$. Then, we substitute the optimum $\hat{\phi}_2^{CDS,\ast}(t,1,0)$ given by (58) into the equation (62). We obtain

\[
\frac{d B(1^{0,0})(t)}{dt} = D_2 \left(t,B(1^{0,0})(t)\right), \quad B(1^{0,0})(T) = 1, \gamma,
\]

where $D_2$ is defined on $(t,B) \in [0,T] \times \mathbb{R}_+$, and given by

\[
D_2(t,B) := K_2 B + U_2(t) B \frac{\gamma}{r - \mu} + A_2(t),
\]
with coefficients
\[
K_2 := 2h_2^0(1, 0) - \gamma h_2(1, 0) - a(\gamma),
\]
\[
A_2(t) := -B^{(1,1)}(t) h_2^0(1, 0),
\]
\[
U_2(t) := \gamma h_2(1, 0) \left( \frac{h_2(1, 0)}{B^{(1,1)}(t) h_2^0(1, 0)} \right)^{\frac{1}{\gamma}} - B^{(1,1)}(t) h_2^0(1, 0) \left( \frac{h_2(1, 0)}{B^{(1,1)}(t) h_2^0(1, 0)} \right)^{\frac{1}{\gamma}}
\]
\[
= (\gamma - 1) h_2(1, 0) \eta_2^{\frac{1}{\gamma}} \left( \frac{h_2(1, 0)}{h_2^0(1, 0)} \right)^{\frac{1}{\gamma}},
\]
where \( \eta_2 = \frac{h_2(1, 0)}{h_2^0(1, 0)} \).

We numerically solve the above nonlinear ODE (unique solution guaranteed by Lemma 4.3), and then plug the corresponding solution into (58) to explicitly obtain the optimal number of shares invested in the CDS asset referencing name “2” after the default of name “1”.

Similarly, after substituting the optimum \( \hat{\phi}_1^{CDS}(t, 0, 1) \) given by (57) into the equation (63), we obtain
\[
\frac{d B^{(0,1)}(t)}{dt} = D_1 \left( t, B^{(0,1)}(t) \right), \quad B^{(0,1)}(T) = \frac{1}{\gamma},
\]
where the function
\[
D_1(t, B) = K_1 B + U_1(t) B^{-\frac{1}{\gamma}} + A_1(t),
\]
with the coefficients
\[
K_1 := 2h_1^0(0, 1) - \gamma h_1(0, 1) - a(\gamma),
\]
\[
A_1(t) := -B^{(1,1)}(t) h_1^0(0, 1),
\]
\[
U_1(t) := \gamma h_1(0, 1) \left( \frac{h_1(0, 1)}{B^{(1,1)}(t) h_1^0(0, 1)} \right)^{\frac{1}{\gamma}} - B^{(1,1)}(t) h_1^0(0, 1) \left( \frac{h_1(0, 1)}{B^{(1,1)}(t) h_1^0(0, 1)} \right)^{\frac{1}{\gamma}}
\]
\[
= (\gamma - 1) h_1(0, 1) \eta_1^{\frac{1}{\gamma}} \left( \frac{h_1(0, 1)}{h_1^0(0, 1)} \right)^{\frac{1}{\gamma}},
\]
where \( \eta_1 = \frac{h_1(0, 1)}{h_1^0(0, 1)} \).

We numerically solve the above nonlinear ODE (unique solution guaranteed by Lemma 4.3), and then plug the corresponding solution into (57) to explicitly obtain the optimal number of shares invested in the CDS asset referencing name “2” after the default of name “1”. Next, we rewrite Eq. (64) as the following quasilinear first-order ODE:
\[
u'(t) = -C(t, \nu(t)), \quad \nu(T) = \frac{1}{\gamma},
\]
where \( t \in [0, T] \), and
\[
C(t, \nu) := \left\{ \gamma \hat{\phi}_1^{CDS}(t, \nu) \left[ (h_1(0, 1) + h_2(0, 0)) \hat{\phi}_1^{L,\nu}(t, 0, 0) - L_1 h_1(0, 0) - \hat{\phi}_1^{L,\nu}(r, 0, 1) h_2(0, 0) \right] + \gamma \hat{\phi}_2^{CDS}(t, \nu) \left[ (h_1(0, 1) + h_2(0, 0)) \hat{\phi}_2^{L,\nu}(t, 0, 0) - L_2 h_2(0, 0) - \hat{\phi}_2^{L,\nu}(t, 1, 0) h_1(0, 0) \right] + a(\gamma) - 2 \left( h_1^0(0, 0) + h_2^0(0, 0) \right) \right\} \nu
\]
\[
+ B^{(1,0)}(t) h_1^0(0, 0) \left\{ 1 + \gamma \hat{\phi}_1^{CDS}(t, \nu)(L_1 - L_1^{(0,0)}(t, 0, 0)) \right\}^\gamma
\]
\[
+ B^{(0,1)}(t) h_2^0(0, 0) \left\{ 1 + \gamma \hat{\phi}_2^{CDS}(t, \nu)(L_2 - L_2^{(0,0)}(t, 0, 0)) \right\}^\gamma
\]
\[
+ B^{(0,1)}(t) h_2^0(0, 0) \left\{ 1 + \gamma \hat{\phi}_1^{CDS}(t, \nu)(L_1 - L_1^{(0,0)}(t, 0, 0)) \right\}^\gamma
\]
\[
+ B^{(0,1)}(t) h_2^0(0, 0) \left\{ 1 + \gamma \hat{\phi}_2^{CDS}(t, \nu)(L_2 - L_2^{(0,0)}(t, 0, 0)) \right\}^\gamma.
\]

We use a fixed point algorithm to solve the coupled system consisting of Eq. (31) and the system of two nonlinear equations given by Eq. (59) and Eq. (60). Notice also that explicit expressions for \( \hat{\phi}_1^{L,\nu}(t, z) \), \( i = 1, 2, \ z \in \{0, 1\}^2 \), can be obtained from Corollary A.2. This algorithm initially sets \( u(t) = \frac{B^{(1,0)}(t) + B^{(0,1)}(t)}{2} \).

Then, it keeps iterating between solving for \( \hat{\phi}_1^{CDS}(t, 0, 0) \) and \( \hat{\phi}_2^{CDS}(t, 0, 0) \) given pre-specified \( u(t) \), and the ODE (31) given the latest estimates of \( \hat{\phi}_1^{CDS}(t, 0, 0) \) and \( \hat{\phi}_2^{CDS}(t, 0, 0) \). It stops when a desired level of convergence is achieved.
References


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