

# Projects and Team Dynamics

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## Abstract

This paper studies the dynamic collaboration of a team on a project that progresses gradually over time and generates a payoff upon completion. The main result is that members of a larger team work harder than members of a smaller team if and only if the project is sufficiently far from completion. In contrast, as the project gets close to completion, the aggregate effort of a larger team can become less than that of a smaller team due to aggravated free-riding. This result provides a rationale for the formation of project teams even without mutual monitoring, peer pressure, synergies, or non-pecuniary benefits from teamwork. In addition, this result has three implications in the organization of partnerships and when a manager recruits agents into a team to undertake a project on her behalf. First, given a fixed budget, larger teams are preferable if the project is large. Second, the manager can benefit from dynamically decreasing the team size as the project approaches completion. Third, smaller teams and asymmetric compensation are preferable if the project is small.

Keywords: Projects, moral hazard in teams, team formation, partnerships, differential games.

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# 1 Introduction

Teamwork is central in the organization of firms and partnerships. Between 1987 and 1996, the use of employee participation teams nearly doubled from 37% to 66% among Fortune 1000 corporations (Lawler, Mohrman and Benson (2001)).<sup>1</sup> Despite the theoretical predictions that effort and group size should be inversely related (Olson (1965), Andreoni (1988), Bonatti and Hörner (2011) and others), empirical studies commonly find that organizing workers into teams or providing group incentives has increased productivity in both manufacturing and service firms. Hamilton, Nickerson and Owan (2003) find that the adoption of teamwork and group incentives improved worker productivity for apparel production, as is the case for Continental airlines (Knez and Simester (2001)), steel finishing lines (Boning, Ichniowski and Shaw (2007)), and call centers (Batt (1999)).

To explain why the adoption of teamwork often leads to increased productivity in organizations, in spite of the free-rider problem, scholars have argued that teams benefit from various motivational forces such as complementary skills (Lazear (1998)), peer pressure to achieve a group norm (Kandel and Lazear (1992)), *warm-glow* (Andreoni (1990)), and non-pecuniary benefits such as more engaging work and social interaction.

I develop a tractable framework to study the team problem faced by a group of agents who collaborate to complete a project. The primary focus is on how the agents' incentives depend on the team composition and on how far the project is from completion. Using this framework, I examine (i) how the agents should organize into a partnership, and (ii) how a manager who recruits agents into a team to carry out a project on her behalf, should determine the team composition as well as the agents' compensation scheme.

The key features of the model are that the project progresses gradually and stochastically towards completion at a rate that depends on the agents' costly effort, and it generates a payoff when it is completed. Many applications fall within this framework. For instance, consider new product development, where a group of individuals collaborate on the design and manufacture of the product: the project progresses gradually, and it starts generating a revenue stream after it is released to the market. Start-up companies also share these dynamics: their evolution is uncertain, and they (predominantly) generate value for the stakeholders when they are acquired by a larger corporation or they become public. Similarly,

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<sup>1</sup>Since the late 1990s, team use seems to have reached a plateau, but it's a relatively high plateau (Lazear and Shaw (2007)).

these features are common in many consulting, marketing, as well as construction projects.

One of the contributions of this paper is that it provides a rationale for the formation of project teams even without mutual monitoring, peer pressure, synergies, or non-pecuniary benefits from teamwork.

## Outline of the Results

A Markov Perfect equilibrium is characterized by a system of ordinary differential equations subject to a set of boundary conditions. By examining how the geometry of the solution depends on the parameters of the problem, I obtain insights about how the agents' incentives to exert effort at different stages of the project depend on the team composition (i.e., the team size, each agent's reward, patience level, and effort costs), and on the degree of uncertainty associated with the evolution of the project.<sup>2</sup> A key result is that agents increase their effort as the project progresses. Intuitively, because they discount time and they are compensated upon completion, they have stronger incentives the closer the project is to completion. This result was first shown by Kessing (2007) who studied a similar model, and its implication is that efforts are strategic complements in this model. This is because by increasing his effort level, an agent brings the project closer to completion, which incentivizes others to also increase their effort.

The main result is that members of a larger team work harder than members of a smaller team - both individually and on aggregate - if and only if the project is sufficiently far from completion.<sup>3</sup> Intuitively, by increasing the size of the team, agents obtain stronger incentives to free-ride. However, because the total progress that needs to be carried out is fixed, the agents benefit from the ability to complete the project sooner, which increases the present discounted value of their reward, and consequently strengthens their incentives. Let us refer to these forces as the *free-riding* and the *encouragement effect*, respectively.<sup>4</sup> Because the marginal cost of effort is increasing and agents work harder the closer the project is to completion, their incentives to free-ride, and consequently the free-riding effect, becomes stronger as the project progresses. On the other hand, the benefit of being able to *speed up* the project

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<sup>2</sup>A similar approach is used by Cao (2010), who studies a continuous-time version of the patent race of Harris and Vickers (1985).

<sup>3</sup>This result holds both if the project is a public good so that each agent's reward is independent of the team size, and if the project generates a fixed payoff that is shared among the team members.

<sup>4</sup>The latter is reminiscent to the encouragement effect in Bolton and Harris (1999), which reveals that more experimentation by the other team members in the future increases each agent's present incentives to experiment.

in a bigger team is smaller the less progress remains, and hence the encouragement effect becomes weaker as the project progresses. Therefore, the encouragement effect dominates the free-riding effect, and consequently members of a larger team work harder than those of a smaller team, if and only if the project is sufficiently far from completion.

This result has two implications in the organization of partnerships. First, if the project is a public good so that each agent's reward is independent of the group size, then expanding the partnership *ad-infinitum* is optimal. On the other hand, if the project generates a payoff upon completion that is shared among the team members, then agents prefer to expand the partnership only if the project is sufficiently *large*.<sup>5</sup>

In the latter part of the paper, I introduce a manager who recruits a group of agents to undertake a project on her behalf. A key result is that with a symmetric compensation scheme, rewarding the agents only upon completion is optimal. The intuition is that by backloading payments (as compared to rewarding the agents for reaching intermediate milestones), the manager can provide the same incentives at the early stages of the project, while providing stronger incentives when the project is close to completion.<sup>6</sup>

These results have three implications with respect to team recruiting. First, larger teams are preferable if the project is sufficiently large. To see the intuition behind this result, recall that a larger team works harder relative to smaller one if and only if the project is sufficiently far from completion. Because the team size is chosen before the agents begin to work, the benefit from a larger team working harder while the project is far from completion outweighs the loss from working less when it is close to completion only if it is sufficiently large.

Second, a manager can benefit from dynamically decreasing the size of the team as the project gets close to completion. The intuition is that she prefers a larger team while the project is far from completion since free-riding is not a major concern, while she prefers a smaller team when the project gets close to completion. With two agents, this can be implemented using an asymmetric compensation scheme in which one agent receives a reward as soon as the project hits a pre-specified intermediate milestone and no further compensation so that he stops working, while the second agent is rewarded only when the project is completed.

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<sup>5</sup>A project is referred to as *large* if the expected amount of progress necessary to complete it is large.

<sup>6</sup>If asymmetric rewards are permitted, then I show that compensating the agents for reaching intermediate milestones can be beneficial, as it enables the manager to effectively change the team size dynamically as the project progresses.

Finally, I show that an asymmetric compensation scheme can be beneficial even if the manager compensates both agents when the project is completed. In particular, with two identical agents, the manager is better off rewarding them asymmetrically if the project is sufficiently small, because the agent who receives the larger reward can *rely* less on the other agent (to exert effort), which mitigates the free-rider problem.

## Related Literature

First, this paper is related to the moral hazard in teams literature (Holmström (1982), Ma, Moore and Turnbull (1988), Bagnoli and Lipman (1989), Legros and Matthews (1993) and others). These papers focus on the free-rider problem, which arises when each agent must share the benefit of his effort with the other members of the team, and they explore ways to restore efficiency.

Most closely related to this paper is the literature on dynamic contribution games. The general theme of these papers is that a group of agents interact repeatedly, and in every period (or moment) they choose their contribution to the public good at a personal cost, which provides a benefit to the entire group. This literature can be broadly classified into two camps.

The first, which this paper belongs to, comprises of papers that study games in which contributions accumulate, and a payoff is generated when the total contributions reach a certain threshold. Admati and Perry (1991) consider a setting in which two agents take turns in contributing to a public good, and they characterize an equilibrium where agents make small contributions at a time, each conditional on the previous contributions of the other agent. Their main result is that socially desirable projects may be not *completed* due to the free-rider problem.<sup>7</sup> Marx and Matthews (2000) consider a simultaneous action  $n$  – player game, and they show that multiple contribution periods can achieve a higher provision level of the public good than what can be achieved in the single period setting provided that there is a discrete payoff jump upon completion of the project.<sup>8</sup> Yildirim (2006) and Kessing (2007) study models related to the one in this paper, and they show

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<sup>7</sup>A related problem is studied by Compte and Jehiel (2004), who allow for the possibility that an agent terminates the game early, and collects a (history dependent) outside option. Their main result is that the presence of such an outside option gives rise to gradualism, because each agent holds back contributing to prevent the other agent from terminating the game early.

<sup>8</sup>Duffy, Ochs and Vesterlund (2007) experimentally test this prediction. They find that contributions in the repeated game are higher than in the static game, but the increase does not depend crucially on the existence of a discrete payoff jump upon completion.

that if the project generates a payoff only when it is completed, then contributions become strategic complements even though there are no complementarities in the agents' production function.<sup>9</sup> In addition, Yildirim (2006) shows that individual effort decreases in the team size when the project is near completion, while effort increases in the team size at the early stages of the project provided that it is sufficiently large.

The second camp comprises of models in which agents receive flow payoffs throughout the duration of the game. Fershtman and Nitzan (1991) study a differential game in which at every moment, each agent's flow payoff is a function of the accumulated contributions. Their main result is that in a Markovian equilibrium, each agent's contribution is a decreasing function of the total contribution, which implies that contributions are strategic substitutes. Recently, Battaglini, Nunnari and Palfrey (2012) study a related game under more general assumptions, and they show that it has a continuum of equilibria, some of which exhibit strategic complementarity and some strategic substitutability. Lockwood and Thomas (2002) and Matthews (2011) also study related contribution games.

I make the following contributions to this literature. First, I propose a tractable framework to analyze the dynamic problem faced by a group of agents who collaborate over time to complete a project, as well as the problem faced by a manager who needs to determine the team composition and how to best compensate the agents. This framework can be useful for addressing an array of other questions related to dynamic moral hazard problems that involve completing a project. Second, contrary to previous literature, while mutual monitoring, synergies, and *warm-glow* are helpful for explaining the benefits of teamwork, I show that they are actually not necessary when the team's objective is to complete a project. Moreover, when compared to the results of Bonatti and Hörner (2011) who establish an inverse relationship between team size and aggregate effort, this paper can explain why the adoption of teams is more prevalent in manufacturing than in service operations (Batt (1999)), where tasks have more of a breakthrough feature (e.g., call centers).<sup>10</sup> Third, as discussed earlier, I draw several insights pertaining to the organization of partnerships, as well as the formation and compensation of project teams.

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<sup>9</sup>A unique feature of the model of Yildirim (2006) is that the project comprises of multiple discrete stages, and in every period, the current stage is completed as long as at least one agent exerts effort (which is binary). As a result, strategies are mixed in a symmetric equilibrium, and higher effort is the interpretation that each agent is more likely to exert effort.

<sup>10</sup>While Bonatti and Hörner (2011) focus on the uncertainty pertaining to the feasibility of the project, their result that the aggregate effort of the team decreases in its size (equation 7) continues to hold as  $\bar{p} \rightarrow 1$ , in which case the project is known to be feasible.

The remainder of this paper is organized as follows. Section 2 introduces the model. Section 3 characterizes the equilibria of the game, and establishes some basic results. Section 4 examines how the size of the team influences the agents' incentives, and the implications of this result in the organization of partnerships. Section 5 studies the problem faced by a manager who recruits agents into a team to undertake a project, and she must determine the team composition and how to compensate the team members. Finally, Section 6 concludes. Appendix A contains a discussion of equilibria with non-Markovian strategies and a robustness test of the main result. All proofs are provided in Appendix B.

## 2 The Model

A team of  $n$  agents collaborate to complete a project. Time  $t \in [0, \infty)$  is continuous. The project starts at some initial state  $q_0 < 0$ , its state  $q_t$  evolves according to a stochastic process, and it is completed at the first time  $\tau$  such that  $q_t$  hits the completion state which is normalized to 0. Note that  $|q_0|$  is interpreted as the size of the project. Agent  $i \in \{1, \dots, n\}$  is risk neutral, discounts time at rate  $r > 0$ , and receives a pre-specified reward  $V_i > 0$  upon completing the project. An incomplete project has zero value, and each agent's outside option equals 0. At time  $t$  each agent observes the state of the project  $q_t$ , and exerts costly effort to influence the drift of the stochastic process

$$dq_t = \left( \sum_{i=1}^n a_{i,t} \right) dt + \sigma dW_t,$$

where  $a_{i,t}$  denotes the effort level of agent  $i$  at time  $t$ ,  $\sigma > 0$  captures the degree of uncertainty associated with the evolution of the project, and  $W_t$  is a standard Brownian motion.<sup>11,12</sup> Efforts are unobservable and each agent's flow cost of exerting effort  $a$  is given by  $c(a) = \frac{\lambda}{p+1} a^{p+1}$ , where  $\lambda > 0$  and  $p \geq 1$ .

At every moment  $t$ , agent  $i$  observes the state of the project  $q_t$ , and chooses his effort strategy  $A_{i,t} = \{a_{i,s}\}_{s \geq t}$  to maximize his expected discounted payoff while taking into account

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<sup>11</sup>Note that  $q_t$  will hit the completion state at some finite time with probability 1 even if no agent ever exerts any effort. This *peculiarity* can be avoided if effort influences both the drift and the diffusion of the stochastic process such that  $dq_t = 0$  if  $a_{i,t} = 0$  for all  $i$ . While the analysis of this case is intractable, numerical analysis with  $dq_t = \left( \sum_{i=1}^n a_{i,t} \right) dt + \sigma \left( \sum_{i=1}^n a_{i,t} \right)^{1/2} dW_t$  suggests that the main result (i.e., Theorem 2), and its implications continue to hold.

<sup>12</sup>I assume that efforts are perfect substitutes. To capture the notion that agents are more productive when working in teams (due to complementary skills), one can consider the case in which the project evolves according to  $dq_t = \left( \sum_{i=1}^n a_{i,t}^{1/\gamma} \right)^\gamma dt + \sigma dW_t$ , where  $\gamma \geq 1$ . The main result (i.e., Theorem 2), and its implications continue to hold.

the effort strategies  $A_{-i,t} = \{a_{-i,s}\}_{s \geq t}$  of the other team members. As such, his expected discounted payoff function satisfies

$$J_i(q_t) = \max_{A_{i,t}} \mathbb{E}_\tau \left[ e^{-r(\tau-t)} V_i - \int_t^\tau e^{-r(s-t)} c(a_{i,s}) ds \mid A_{-i,t} \right], \quad (1)$$

where the expectation is taken with respect to  $\tau$ : the random variable that denotes the completion time of the project.

Assuming that  $J_i(\cdot)$  is twice differentiable for all  $i$ , and using standard arguments (Dixit (1999)), one can derive the Hamilton-Jacobi-Bellman (hereafter HJB) equation for the expected discounted payoff function for agent  $i$ :

$$rJ_i(q_t) = \max_{A_{i,t} | A_{-i,t}} \left\{ -c(a_{i,t}) + \left( \sum_{j=1}^n a_{j,t} \right) J'_i(q_t) + \frac{\sigma^2}{2} J''_i(q_t) \right\} \quad (2)$$

defined on  $(-\infty, 0]$  subject to the value-matching conditions

$$\lim_{q \rightarrow -\infty} J_i(q) = 0 \quad \text{and} \quad J_i(0) = V_i. \quad (3)$$

(2) asserts that agent  $i$ 's flow payoff is equal to his flow cost of effort, plus his marginal benefit from bringing the project closer to completion times the aggregate effort of the team, plus a term that captures the sensitivity of his payoff to the volatility of the project. To interpret (3), observe that as  $q \rightarrow -\infty$ , the expected time until the project is completed so that agent  $i$  collects his reward diverges to  $\infty$ . Because  $r > 0$ , his expected discounted payoff asymptotes to 0. On the other hand, because he receives his reward and exerts no further effort after the project is completed,  $J_i(0) = V_i$ .<sup>13</sup>

Finally, observe that  $V_i$  and  $\lambda$  are isomorphic. Therefore, without loss of generality, for the remainder of this paper I normalize  $\lambda = 1$ .<sup>14</sup>

<sup>13</sup>By noting that  $J_i(q) \in [0, V_i]$  for all  $q$  and  $i$ , it follows that the transversality condition  $\lim_{t \rightarrow \infty} \mathbb{E}[e^{-rt} J_i(q_t)] = 0$  of the verification theorem (p. 123 in Chang (2004)) is satisfied, thus ensuring that a solution to the system of HJB equations (2) subject to (3) is indeed optimal for (1).

<sup>14</sup>To verify this, let  $\tilde{J}_i(q) = \frac{J_i(q)}{\lambda}$ , substitute this into (2), and observe that  $\lambda$  cancels out. Using (3), observe that  $\tilde{J}_i(\cdot)$  satisfies  $\lim_{q \rightarrow -\infty} \tilde{J}_i(q) = 0$  and  $\tilde{J}_i(0) = \frac{V_i}{\lambda}$ .

### 3 Results

In Section 3.1, I characterize the Markov Perfect equilibrium of the game (hereafter MPE) and I discuss the possibility that the agents adopt non-Markovian strategies. Section 3.2 examines how the agents' incentives depend on the parameters of the problem. Then in Section 3.3, I examine the first-best outcome of the game and I compare it to the MPE.

#### 3.1 Markov Perfect Equilibrium

I assume that strategies are Markovian, so that at every moment, each agent chooses his effort level as a function of the current state of the project. Therefore, given  $q$ , agent  $i$  chooses his effort level  $a_i(q)$  such that

$$a_i(q) \in \arg \max_{a_i} \{a_i J'_i(q) - c(a_i)\} .$$

The first-order condition for agent  $i$ 's problem is  $J'_i(q) = c'(a_i)$ : each agent chooses his effort level such that the marginal cost of effort is equal to the marginal benefit associated with bringing the project closer to completion. Because  $c'(0) = 0$  and  $c(\cdot)$  is strictly increasing, given any  $q$  there exists a unique non-negative effort level  $a_i(q)$  that satisfies the first-order condition as long as  $J'_i(q) \geq 0$ . Suppose for now that  $J'_i(q) \geq 0$  for all  $q$ , and let  $f(\cdot) = c'^{-1}(\cdot)$ .<sup>15</sup> Then  $a_i(q) = f(J'_i(q))$ , and by substituting this into (2), the expected discounted payoff for agent  $i$  satisfies

$$rJ_i(q) = -c(f(J'_i(q))) + \left[ \sum_{j=1}^n f(J'_j(q)) \right] J'_i(q) + \frac{\sigma^2}{2} J''_i(q) \quad (4)$$

subject to (3).

A MPE is characterized by the system of nonlinear ordinary differential equations defined by (4) subject to (3) for all  $i \in \{1, \dots, n\}$ . As a result, to show that a MPE exists, it suffices to show that a solution to the system of differential equations exists, and  $J'_i(q) \geq 0$  for all  $i$  and  $q$ . The MPE will be unique if the system of differential equations has exactly one solution, because every MPE must satisfy (4) subject to (3), and the first-order condition is both necessary and sufficient.

**Theorem 1.** *A Markov Perfect equilibrium (MPE) for the game defined by (1) exists. For*

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<sup>15</sup>Theorem 1 establishes that in fact  $J'_i(q) > 0$  for all  $q$ , which implies that the first-order always binds. The interpretation is that each agent is strictly better off the closer the project is to completion, and it implies that effort is non-negative at every state of the project.

each agent  $i$ , the expected discounted payoff function  $J_i(q)$  is infinitely differentiable on  $(-\infty, 0]$ , and it satisfies:

(i)  $0 < J_i(q) \leq V_i$  for all  $q$ .

(ii)  $J'_i(q) > 0$  for all  $q$ , and hence the equilibrium effort  $a_i(q) > 0$  for all  $q$ .

(iii)  $J''_i(q) > 0$  for all  $q$ , and hence  $a'_i(q) > 0$  for all  $q$ .

If agents are symmetric, then the MPE is symmetric. Moreover, the equilibrium is unique with  $n$  symmetric agents or 2 asymmetric agents.<sup>16</sup>

$J'_i(q) > 0$  implies that each agent is strictly better off, the closer the project is to completion. Because  $c'(0) = 0$  (i.e., the marginal cost of a *little* effort is negligible), each agent exerts a strictly positive amount of effort at every state of the project:  $a_i(q) > 0$  for all  $q$ .<sup>17</sup>

The facts that agents are impatient, they incur the cost of effort at the time effort is exerted, and they are compensated upon completing the project implies that they have stronger incentives the closer the project is to completion:  $a'_i(q) > 0$  for all  $q$ . The implication of this result is that efforts are strategic complements in this model. This is because a higher effort level by one agent brings the project closer to completion, which incentivizes the other agents to also raise their effort. This result was first shown by Yildirim (2006) and Kessing (2007), and it is in contrast to static models (Holmström (1982)), dynamic models in which the agents receive flow payoffs while the project is in progress (Fershtman and Nitzan (1991)), as well as dynamic models in which the project can be completed instantaneously (Bonatti and Hörner (2011)) where efforts are strategic substitutes.

This analysis raises the question whether agents can increase their expected discounted payoff by adopting non-Markovian strategies, so that their effort at  $t$  depends on the entire evolution path of the project  $\{q_s\}_{s \leq t}$ . While a formal analysis of this case is beyond the scope of this paper, following Sannikov and Skrzypacz (2007), I conjecture that there does not exist a symmetric Public Perfect equilibrium in which agents can achieve a higher expected discounted payoff than the MPE at any state of the project. See Appendix A.1 for details.

*Remark 1.* If there is an exogenous *cancellation* state  $Q_c < q_0$  such that the agents abandon the project and receive payoff 0 at the first time that  $q_t$  hits  $Q_c$ , then effort needs no longer be increasing in  $q$ . Instead, it increases in  $q$  only if  $Q_c$  is sufficiently small, it is U-shaped if

<sup>16</sup>To simplify notation, if the agents are symmetric, then the subscript  $i$  is interchanged with the subscript  $n$  to denote the team size throughout the remainder of this paper.

<sup>17</sup>If  $c'(0) > 0$ , then there exists a quitting threshold  $Q_q$ , such that agent  $i$  exerts 0 effort on  $(-\infty, Q_q]$ , while he exerts strictly positive effort on  $(Q_q, 0]$ , and his effort increases in  $q$ .

$Q_c$  is in some medium range, and it decreases in  $q$  if  $Q_c$  is sufficiently close to 0. Intuitively, agents have incentives to work harder when the state of the project is near the cancellation state to avoid hitting it, and these incentives are stronger, the larger  $Q_c$  is. However, if  $Q_c$  is endogenous, then agents are always better off choosing  $Q_c^* = -\infty$ .

On the other hand, if agents receive a strictly positive outside option at the first time that  $q_t$  hits  $Q_c$ , then in a team comprising of symmetric agents, the optimal cancellation state  $Q_c^*$  satisfies the smooth-pasting condition  $J'_n(Q_c^*) = 0$ , and effort is increasing in  $q$ .

### 3.2 Comparative Statics

This section establishes some comparative statics, which are helpful for understanding how the agents' incentives depend on the parameters of the problem. In particular, to examine the effect of each parameter to the agents' incentives, I consider two symmetric teams that differ in exactly one attribute: their members' rewards  $V_i$ , patience levels  $r_i$ , or the volatility of the project  $\sigma$ .

**Proposition 1.** *Consider two teams comprising of symmetric agents.*

- (i) *If  $V_1 < V_2$ , then other things equal,  $a_1(q) < a_2(q)$  for all  $q$  ,<sup>18</sup>*
- (ii) *If  $r_1 > r_2$ , then other things equal, there exists an interior threshold  $\Theta_r$  such that  $a_1(q) \leq a_2(q)$  if and only if  $q \leq \Theta_r$  ; and*
- (iii) *If  $\sigma_1 > \sigma_2$ , then other things equal, there exist interior thresholds  $\Theta_{\sigma,1} \leq \Theta_{\sigma,2}$  such that  $a_1(q) \geq a_2(q)$  if  $q \leq \Theta_{\sigma,1}$  and  $a_1(q) \leq a_2(q)$  if  $q \geq \Theta_{\sigma,2}$  .*

The intuition behind statements (i) is straightforward. If the agents receive a bigger reward, then they always work harder in equilibrium.

Statement (ii) asserts that a team of less patients agent works harder relative to a team of more patient agents if and only if the project is sufficiently close to completion. Intuitively, less patient agents have more to gain from an earlier completion (provided that the project is sufficiently close to completion). However, bringing the completion time forward requires them to exert more effort, whose costs are incurred at the time that effort is exerted, whereas the reward is only collected upon completion of the project. Therefore, the benefit from bringing the completion time forward (by exerting more effort) outweighs its cost only when the project is sufficiently close to completion.

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<sup>18</sup>Since the teams differ in a single parameter (i.e., their patience level  $V_i$  in statement (i)), abusing notation, I let  $a_i(\cdot)$  denote each agent's effort strategy corresponding to the parameter with subscript  $i$ .

Finally, statement (iii) asserts that incentives become stronger in the volatility of the project  $\sigma$  when it is far from completion, while the opposite is true when it gets close to completion.<sup>19</sup> To see the intuition behind this result, first note that as the volatility increases, it becomes *more likely* that the project will be completed either earlier than expected (*upside*), or later than expected (*downside*). If the project is sufficiently far from completion, then  $J_i(q) \simeq 0$  so that the downside is negligible, while  $J_i''(q) > 0$  implies that the upside is not (negligible), and consequently  $a_1(q) \geq a_2(q)$ . On the other hand, because the completion time of the project is non-negative, the upside diminishes as the project approaches completion. Therefore, when the project is sufficiently close to completion (i.e.,  $q \geq \Theta_{\sigma,2}$ ), the downside is bigger than the upside so that  $a_1(q) \leq a_2(q)$ .

### 3.3 First-Best Outcome

To obtain a benchmark for the agents' equilibrium effort levels, I compare them to the first-best outcome, where at every moment, each agent chooses his effort level to maximize the team's, as opposed to his individual expected discounted payoff. I focus on the symmetric case, and denote by  $\hat{J}_n(q)$  and  $\hat{a}_n(q)$  the first-best expected discounted payoff and effort level of each member of an  $n$ -person team, respectively. The first-order condition for each agent's effort level satisfies  $\hat{a}_n(q) \in \arg \max_a \{an\hat{J}'_n(q) - c(a)\}$ , and substituting the first order condition into (2) yields

$$r\hat{J}_n(q) = -c\left(f\left(n\hat{J}'_n(q)\right)\right) + nf\left(n\hat{J}'_n(q)\right)\hat{J}'_n(q) + \frac{\sigma^2}{2}\hat{J}''_n(q)$$

subject to the boundary conditions (3). It is straight-forward to show that the properties established in Theorem 1 apply for  $\hat{J}_n(q)$  and  $\hat{a}_n(q)$ . In particular, the system of first-best ODE subject to the boundary conditions (3) has a unique solution,  $\hat{J}'_n(q) > 0$  for all  $q$  so that the first order condition always binds, and  $\hat{J}''_n(q) > 0$  for all  $q$ , which implies that  $\hat{a}'_n(q) > 0$ ; i.e., similar to the MPE, first-best effort increases in the state of the project.

**Proposition 2.** *In a team of  $n \geq 2$  agents,  $\hat{a}_n(q) > a_n(q)$  and  $\hat{J}_n(q) > J_n(q)$  for all  $q$ .*

This result is not surprising: due to the free-rider problem, in the MPE, each agent exerts strictly less effort and he is strictly worse off at every state of the project as compared to the case in which agents behave collectively by choosing their effort level at every moment to maximize the team's expected discounted payoff.

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<sup>19</sup>A limitation of this result is that it does not guarantee that  $\Theta_{\sigma,1} = \Theta_{\sigma,2}$ , which implies that it does not provide any prediction about how the agents' effort depends on  $\sigma$  when  $q \in [\Theta_{\sigma,1}, \Theta_{\sigma,2}]$ . However, numerical analysis indicates that in fact  $\Theta_{\sigma,1} = \Theta_{\sigma,2}$ .

## 4 Team Dynamics

This Section contains the main result of the paper, Theorem 2, which investigates how the individual effort of each agent, as well as the aggregate effort of the team depends on its size. This result will be useful for examining how agents should organize into a partnership in Section 4.2, and how a manager should recruit agents into a team in Section 5. Throughout this Section I assume that agents are symmetric.

### 4.1 The Effect of Team Size to the Agents' Incentives

When examining the relationship between the agents' incentives and the size of the team, it is important to consider how each agent's reward depends on the team size. I consider two cases: the *public good allocation* scheme, wherein each agent receives a reward  $V$  upon completing the project, which does not depend on the size of the team, and the *budget allocation* scheme, wherein each agent receives a reward  $\frac{V}{n}$  upon completing the project.<sup>20</sup>

With  $n$  symmetric agents, each agent's expected discounted payoff function satisfies

$$rJ_n(q) = -c(f(J'_n(q))) + nf(J'_n(q))J'_n(q) + \frac{\sigma^2}{2}J''_n(q)$$

subject to  $\lim_{q \rightarrow -\infty} J_n(q) = 0$  and  $J_n(0) = V_n$ , where  $V_n = V$  or  $V_n = \frac{V}{n}$  under the public good or the budget allocation scheme, respectively.

**Theorem 2.** *Consider two teams comprising of  $n$  and  $m > n$  identical agents. Under both allocation schemes, there exist thresholds  $\Theta_{n,m} < 0$  and  $\Phi_{n,m} \leq 0$  such that*

**(A)**  $a_m(q) \geq a_n(q)$  if and only if  $q \leq \Theta_{n,m}$ ; and

**(B)**  $ma_m(q) \geq na_n(q)$  if and only if  $q \leq \Phi_{n,m}$ .

Statement (A) asserts that under both allocation schemes, members of a larger team work harder than members of a smaller team if and only if the project is sufficiently far from completion. Figure 1 illustrates an example. To see the intuition behind this result, note that by increasing the size of the team, two forces influence the agents' incentives: First, agents obtain stronger incentives to free-ride. To see why, consider an agent's dilemma at time  $t$  to (unilaterally) reduce his effort by a *small* amount  $\varepsilon$  for a *short* interval  $\Delta$ . By doing so, he saves approximately  $\varepsilon c'(a_t) \Delta$  in cost of effort, but at  $t + \Delta$ , the project is (on

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<sup>20</sup>The public good allocation scheme is applicable to cases in which each member of the team receives reputation benefits associated with the completion of a project, or utility from the construction of an animal shelter or a public park. On the other hand, the budget allocation scheme is applicable to cases in which the project generates a monetary reward that is shared among the team members.

expectation)  $\varepsilon\Delta$  father from completion (compared to the scenario in which he does not reduce his effort). In equilibrium, this agent will carry out only  $\frac{1}{n}$  of that *lost* progress, which implies that the benefit from shirking increases in the size of the team. On the other hand, because the total progress that needs to be carried out is fixed, increasing the team size (and holding strategies fixed) implies that the project will (on expectation) be completed sooner. This increases the present discounted value of each agent's reward (i.e.,  $\mathbb{E}_\tau [e^{-r\tau}]$ ), which strengthens his incentives. Therefore, in a larger team, agents benefit from the ability to complete the project sooner. I shall refer to these forces as the *free-riding* and the *encouragement effect*, respectively, and the intuition will follow from examining how the magnitude of these effects changes as the project progresses.

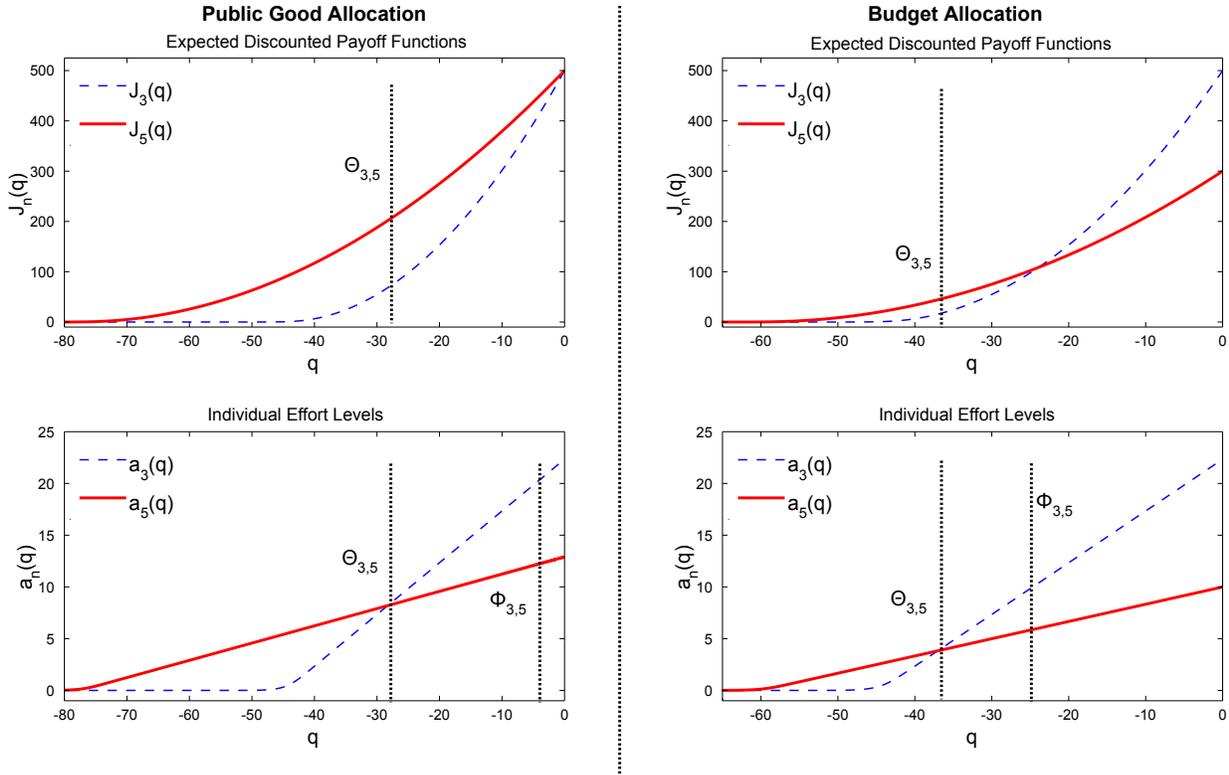


Figure 1: **Illustration of Theorem 2.** The upper panels illustrate each agent's expected discounted payoff under public good (left) and budget (right) allocation for two different team sizes:  $n = 3$  and 5. The lower panels illustrate each agent's equilibrium effort. In both cases, there exists an interior threshold  $\Theta_{3,5}$  such that each member of the larger team exerts more effort relative to each member of the smaller team if and only if  $q \leq \Theta_{3,5}$ . Similarly, there exists an interior threshold  $\Phi_{3,5}$  such that the total aggregate of the larger team is greater than that of the smaller team if and only if  $q \leq \Phi_{3,5}$ .

First, let us consider the free-riding effect, and recall that agents work harder, the closer the project is to completion. By noting that the marginal cost of effort is increasing, it follows that an agent's gain from free-riding, which is proportional to  $c'(a_t)$ , increases in  $q$ .

Therefore, the free-riding effect becomes stronger in the state of the project. In addition, because effort vanishes as  $q \rightarrow -\infty$  in equilibrium and  $c'(0) = 0$ , the free-riding effect is negligible when the project is sufficiently far from completion.

To understand how the magnitude of the encouragement effect changes as the project progresses, it is helpful to consider the deterministic case in which  $\sigma = 0$ . Let  $\tau$  denote the completion time when the team comprises of  $n$  agents, and note that each agent's marginal benefit of bringing the completion time forward is  $-\frac{d}{d\tau}V_n e^{-r\tau} = rV_n e^{-r\tau}$ . Increasing the team size to  $m$  and holding strategies fixed reduces the completion time to  $\frac{n}{m}\tau$ , and each agent's respective marginal benefit now becomes  $rV_m e^{-r\frac{n}{m}\tau}$ . Therefore, the magnitude of the encouragement effect can be measured by the ratio of the marginal benefits:  $\frac{V_m}{V_n} e^{\frac{r(m-n)\tau}{mn}}$ . Under the public good allocation (so that  $\frac{V_m}{V_n} = 1$ ), this ratio increases in  $\tau$  and it is always greater than 1, which implies that the benefit of a bigger team being able to complete the project sooner decreases as the project progresses, and it becomes negligible as the project nears completion. On the other hand, under the budget allocation (in which case  $\frac{V_m}{V_n} = \frac{n}{m}$ ), while the ratio in consideration still increases in  $\tau$ , it is greater than 1 only if  $\tau$  is sufficiently large. Therefore, the encouragement effect is positive when the project is sufficiently far from completion, it becomes weaker as the project progresses, and it is negative when the project is close to completion.

Therefore, if the project is sufficiently far from completion, then the encouragement effect dominates the free-riding effect, and consequently, members of a larger team work harder than those of a smaller team. Conversely, the opposite is true when the project is close to completion.

Statement (B) shows that under both allocation schemes, the aggregate effort exerted by the larger team is greater than that of the smaller team if and only if the project is sufficiently far from completion. When the project is far from completion such that  $q \leq \Theta_{n,m}$ , it is straightforward that the aggregate effort of the larger team exceeds that of the smaller team by statement (A). The perhaps surprising aspect of this result is that the free-riding effect can become so aggravated when the project is near completion, that not only each member of the larger team exerts less effort relative to each member of the smaller team, but also the aggregate effort of the larger team becomes less than that of the smaller team.

By using the same proof technique, one can show that under both allocation schemes, the first-best aggregate effort increases in the team size at every state of the project. This so-

validates the intuition that statement (B) is a consequence of the free-riding effect becoming overwhelmingly stronger than the encouragement effect when the project is close to completion.

Theorem 2 reaches an opposite conclusion relative to earlier results in the moral hazard in teams and the public good contribution literatures that establish an inverse relationship between individual effort (or contribution) and team size (Holmström (1982), Andreoni (1988) and Bonatti and Hörner (2011)). The key difference is that efforts are strategic complements in the model studied in this paper. Therefore as the team size increases, in addition to the free-rider problem becoming aggravated (which is consistent with previous findings), the agents can also benefit from the ability to complete the project sooner.

Note that the thresholds defined in Theorem 2 need not always be interior. In particular, it is possible that  $\Theta_{n,m} = -\infty$  under budget allocation, which would imply that each member of the smaller team always works harder than each member of the larger team. However, numerical analysis indicates that  $\Theta_{n,m}$  is always interior under both allocation schemes. On the other hand,  $\Phi_{n,m}$  is guaranteed to be interior only under budget allocation if effort costs are quadratic, while one can find examples in which  $\Phi_{n,m}$  is interior as well as examples in which  $\Phi_{n,m} = 0$  otherwise. Numerical analysis indicates that the most important parameter that determines whether  $\Phi_{n,m}$  is interior is the convexity of the effort costs, and it is interior as long as effort costs are not too convex (i.e.,  $p$  is sufficiently small). This is intuitive, because more convex effort costs favor the larger team more. In addition, under public good allocation, for  $\Phi_{n,m}$  to be interior, it is also necessary that  $n$  and  $m$  are sufficiently small. Intuitively, this is because the *size of the pie* increases in the team size under this scheme, which (again) favors the larger team.<sup>21</sup>

## 4.2 Partnership Formation

Now let us examine the problem faced by a group of agents who organize into a partnership. Suppose that teams are formed sequentially, and the agents who have already committed to join, decide whether to admit another member.<sup>22</sup> Admission to the team is costless, and no agent will begin to work until the team composition has been finalized.

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<sup>21</sup>The case in which effort costs are linear is examined in Appendix A.2, and an analogous result to Theorem 2 is shown: members of a larger team have stronger incentives relative to those of a smaller team as long as  $n$  is sufficiently small.

<sup>22</sup>Note that because the equilibrium is symmetric, the team members will be in agreement with respect to whether they should admit a new member.

**Proposition 3.** *Suppose that  $n$  identical agents have committed to join a team.*

*(i) Under public good allocation, the team is always better off admitting another member.*

*(ii) Under budget allocation, there exists a threshold  $T_n$  such that the team is better off admitting another member if and only if the project size  $|q_0| \geq T_n$ .*

By adding another member to the team, each agent will need to exert less effort to complete the project, which implies that his total expected discounted effort cost will decrease. If his reward upon completing the project does not depend on the team size, as is the case under public good allocation, then expanding the partnership *ad-infinitum* is optimal.

On the other hand, if each agent who has committed to join must surrender part of his reward in order to expand the team (i.e., under budget allocation), then he will do so only if the gain from being able to complete the project sooner in a bigger group is sufficiently large to offset the decrease in his net payoff upon completing the project. This is true only if the project is sufficiently large. This result is illustrated in the top panels of Figure 1.

## 5 Manager's Problem

In this Section, I introduce a manager who recruits a group of agents to carry out a project on her behalf. Section 5.1 describes the model, and Section 5.2 formulates the manager's problem and establishes some of its properties. Then, Section 5.3 studies her contracting problem: how she should determine the team composition and the agents' compensation scheme to maximize her expected discounted profit.

### 5.1 The Model

The manager is risk neutral, she discounts time at the same rate  $r > 0$  as the agents, and her outside option is normalized to 0. She hires a group of agents to undertake a project on her behalf. The project in consideration has size  $|q_0|$ , and upon completion, it generates a payoff  $U > 0$ . To incentivize the agents, the manager designates a set of milestones  $q_0 < Q_1 < \dots < Q_K = 0$  (where  $K \in \mathbb{N}$ ), and for every  $k \in \{1, \dots, K\}$  she allocates non-negative payments  $\{V_{i,k}\}_{i=1}^n$  that are due upon reaching milestone  $Q_k$  *for the first time*. She then makes a take-it-or-leave-it offer to a group of agents, and once the team composition has been finalized, the agents begin to work.<sup>23,24</sup> For each  $k$ , the manager disburses the

<sup>23</sup>The details of each offer is public information, so that without loss of generality, I can assume that offers will be made such that every one is accepted.

<sup>24</sup>It is important to acknowledge that the manager's contracting space is limited. While a contract a-la-Sannikov (2008) or one in which the agents' payoffs may also depend on time is more desirable, such analysis

payments  $\{V_{i,k}\}_{i=1}^n$  as soon as the project hits  $Q_k$  for the first time. As soon as the project is completed, the manager collects her payoff  $U$ , she disburses the final payments  $\{V_{i,K}\}_{i=1}^n$  to the agents, and the game ends.

Therefore, the manager's problem entails choosing the team size and the agents' contracts to maximize her expected discounted profit at time 0 (i.e., at  $q_0$ ) subject to the agents' incentive compatibility constraints.

## 5.2 A Preliminary Result

I begin by considering the case in which the manager compensates the agents only upon completing the project, and I show in Theorem 3 that her problem is well-defined and it satisfies some desirable properties. Then I explain how this result extends to the case in which the manager also rewards the agents for reaching intermediate milestones.

Given the team size  $n$  and the agents' compensations  $\{V_i\}_{i=1}^n$  that are due upon completion of the project, the manager's expected discounted profit function can be written as

$$F(q) = \mathbb{E}_\tau \left[ e^{-r\tau} \left( U - \sum_{i=1}^n V_i \right) \mid q \right],$$

where the expectation is taken with respect to the project's completion time  $\tau$ , which depends on the agents' strategies and the stochastic evolution of the project.<sup>25</sup> By using the first order condition for each agent's equilibrium effort level as determined in Section 3, the manager's expected discounted profit at any given state of the project satisfies

$$rF(q) = \left[ \sum_{i=1}^n f(J'_i(q)) \right] F'(q) + \frac{\sigma^2}{2} F''(q) \quad (5)$$

defined on  $(-\infty, 0]$  subject to the boundary conditions

$$\lim_{q \rightarrow -\infty} F(q) = 0 \quad \text{and} \quad F(0) = U - \sum_{i=1}^n V_i. \quad (6)$$

The interpretation of these boundary conditions is similar to (3). As the state of the project diverges to  $-\infty$ , its expected completion time diverges to  $\infty$ , and because  $r > 0$ , the manager's expected discounted profit diminishes to 0. On the other hand, the manager's profit is

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does not seem tractable using the present model, and is therefore left for future research.

<sup>25</sup>Note that the subscript  $k$  is dropped when  $K = 1$  (in which case  $Q_1 = 0$ ).

realized when the project is completed, and it equals her payoff  $U$  less the payments  $\sum_{i=1}^n V_i$  disbursed to the agents.<sup>26</sup>

**Theorem 3.** *Given  $(n, \{V_i\}_{i=1}^n)$ , a solution to the manager's problem defined by (5) subject to the boundary conditions (6) and the agents' problem as defined in Theorem 1 exists, and it has the following properties:*

- (i)  $F(q) > 0$  and  $F'(q) > 0$  for all  $q$ .
- (ii)  $F(\cdot)$  is infinitely differentiable on  $(-\infty, 0]$ .

Moreover,  $F(\cdot)$  is unique if the team comprises of  $n$  symmetric or 2 asymmetric agents.

Now let us discuss how Theorems 1 and 3 extend to the case in which the manager rewards the agents upon reaching intermediate milestones. Recall that she can designate a set of milestones, and attach rewards to each milestone that are due as soon as the project reaches the respective milestone for the first time. Let  $J_{i,k}(\cdot)$  denote agent  $i$ 's expected discounted payoff given that the project has reached  $k - 1$  milestones, which is defined on  $(-\infty, Q_k]$ , and note that it satisfies (4) subject to  $\lim_{q \rightarrow -\infty} J_{i,k}(q) = 0$  and  $J_{i,k}(Q_k) = V_{i,k} + J_{i,k+1}(Q_k)$ , where  $J_{i,K+1}(0) = 0$ . The second boundary condition states that upon reaching milestone  $k$ , agent  $i$  receives the reward attached to that milestone, plus the continuation value from future rewards. Starting with  $J_{i,K}(\cdot)$ , it is straightforward that it satisfies the properties of Theorem 1, and in particular, that  $J_{i,K}(Q_{k-1})$  is unique, so that the boundary condition of  $J_{i,K-1}(\cdot)$  at  $Q_{K-1}$  is well-defined. Proceeding backwards, it follows that for every  $k$ ,  $J_{i,k}(\cdot)$  satisfies the properties established in Theorem 1.

To examine the manager's problem, let  $F_k(\cdot)$  denote her expected discounted profit given that the project has reached  $k - 1$  milestones, which is defined on  $(-\infty, Q_k]$ , and note that it satisfies (5) subject to  $\lim_{q \rightarrow -\infty} F_k(q) = 0$  and  $F_k(Q_k) = F_{k+1}(Q_k) - \sum_{i=1}^n V_{i,k}$ , where  $F_{K+1}(Q_k) = U$ . The second boundary condition states that upon reaching milestone  $k$ , the manager receives the continuation value of the project, less the payments that she disburses to the agents for reaching this milestone. Again starting with  $k = K$  and proceeding backwards, it is straightforward that  $F_k(\cdot)$  satisfies the properties established in Theorem 3 for all  $k$ .

### 5.3 Contracting Problem

To begin, Theorem 4 shows that as long as the manager uses a symmetric compensation scheme, she should reward the agents only upon completion of the project.

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<sup>26</sup>Because the manager's outside option is equal to 0, without loss of generality, I can restrict attention to the case in which the payments  $\{V_i\}_{i=1}^n$  are chosen such that  $\sum_{i=1}^n V_i \leq U$ .

**Theorem 4.** *Suppose that the manager employs  $n$  identical agents and she uses a symmetric compensation scheme. Then it is optimal to compensate the agents only upon completion of the project.*

To prove this result, I consider an arbitrary set of milestones and arbitrary rewards attached to each milestone, and I construct an alternative compensation scheme that rewards the agents only upon completing the project and renders the manager better off. Intuitively, because rewards are sunk (in terms of incentivizing the agents) after they are disbursed, by backloading payments, the manager can provide the same incentives at the early stages of the project, while providing stronger incentives when the project is close to completion.

The value of Theorem 4 lies in that it reduces the infinite-dimensional problem of determining the team size, the number of milestones, the set of milestones, and the rewards attached to each milestone into a two-dimensional problem, in which the manager only needs to determine the team size and her budget  $B = \sum_{i=1}^n V_i$  for compensating the agents.

The restriction that the manager compensates the agents symmetrically is not without loss of generality. As shown in Remark 2, an asymmetric scheme that rewards the agents upon reaching (different) intermediate milestones may be desirable, because it enables the manager to dynamically decrease the team size as the project progresses, which in turn mitigates free-riding. However, because individuals value fairness in pay (Baron and Kreps (1999)), it is of interest to examine the symmetric case. The following Proposition examines how the manager should determine her budget.

**Proposition 4.** *Suppose that the manager employs  $n$  identical agents and she compensates them symmetrically. Then her optimal budget  $B$  increases in the projects size  $|q_0|$ .*

When determining her budget, the manager trades off her net profit  $U - B$  and the project's expected completion time  $\tau$ . By increasing her budget, her net profit decreases, but the project's expected completion time also decreases since the agents work harder. Because a larger project takes longer to be completed and the manager discounts time, a decrease in her net profit has a smaller effect on her expected discounted profit at time 0 the larger the project is. Therefore, the benefit from raising the agents' compensations outweighs the decrease in her net profit if and only if the project is sufficiently large.

**Proposition 5.** *Suppose the manager has a fixed budget  $B$  to (symmetrically) compensate a group of identical agents. For any  $m > n$ , there exists a threshold  $T_{n,m}$  such that she is better off employing an  $m$ -member team instead of an  $n$ -member one if and only if the size of the project  $|q_0| \geq T_{n,m}$ .*

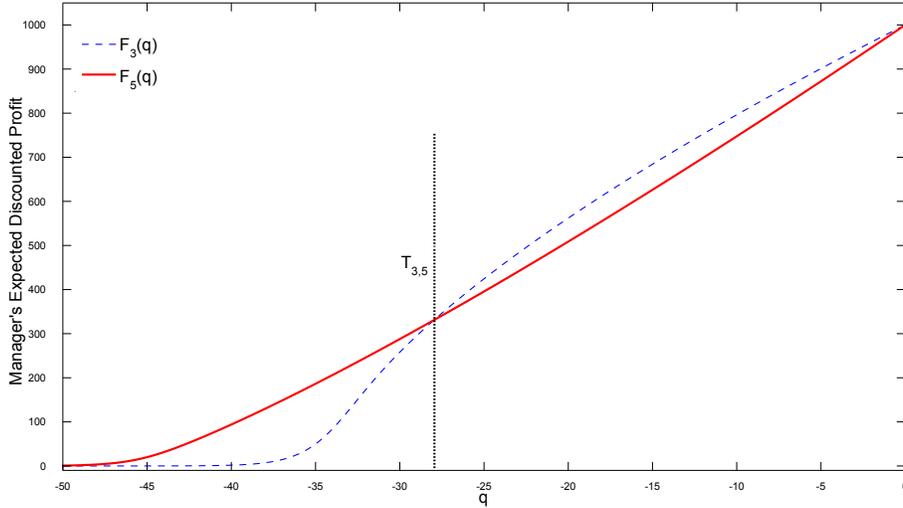


Figure 2: **Illustration of Proposition 5.** Given a fixed budget, the manager’s expected discounted profit is higher if she recruits a 5-member team relative to a 3-member team if and only if the initial state of the project  $q_0$  is to the left of the threshold  $-T_{3,5}$ ; or equivalently, if and only if  $|q_0| \geq T_{3,5}$ .

Given a fixed budget, the manager’s objective is to choose the team size to minimize the expected completion time of the project. This is equivalent to maximizing the aggregate effort of the team along the evolution path of the project. Hence, the intuition behind this result follows from statement (B) of Theorem 2. If the project is small, then on expectation, the aggregate effort of the smaller team will be greater than that of the larger team due to the free-riding effect (on average) dominating the encouragement effect. The opposite is true if the project is large. Figure 2 illustrates an example.

Applying the Monotonicity Theorem of Milgrom and Shannon (1994) leads one to the following Corollary.

**Corollary 1.** *Given a fixed budget to (symmetrically) compensate a group of identical agents, the manager’s optimal team size  $n$  increases in the size of the project  $|q_0|$ .*

The take-away from Proposition 5 (and Corollary 1) is that a larger team is more desirable while the project is far from completion, whereas a smaller team becomes preferable when the project gets close to completion. Therefore, it seems desirable to construct a scheme that dynamically decreases the team size as the project progresses. Suppose that the manager employs two identical agents on a fixed budget  $B$ , and she designates a *retirement state*  $R$ , such that one of the agents is permanently retired (i.e., he stops exerting effort) at the first time that the state of the project hits  $R$ . From that point onwards, the other agent continues

to work alone. Both agents are compensated only upon completion of the project, and the payments (say  $V_1$  and  $V_2$ ) are chosen such that the agents are indifferent with respect to who will retire at  $R$ ; i.e., their expected discounted payoffs are equal at  $q_t = R$ .<sup>27</sup>

**Proposition 6.** *Suppose the manager employs two identical agents with quadratic effort costs. Consider the retirement scheme described above, where the retirement state  $R$  is chosen such that  $|R| \leq \min\{|q_0|, T_{1,2}\}$  and  $T_{1,2}$  is taken from Proposition 5. There exists a threshold  $\Theta_R > |R|$  such that the manager is better off implementing this retirement scheme relative to allowing both agents to work together until the project is completed if and only if its size  $|q_0| < \Theta_R$ .*

First, note that after one agent retires, the other will exert first-best effort until the project is completed. Because the manager's budget is fixed, this retirement scheme is preferable only if it increases the expected aggregate effort of the team along the evolution path of the project. A key part of the proof involves showing that agents have weaker incentives before one of them is retired as compared to the case in which they always work together (i.e., when a retirement scheme is not used). Therefore, the benefit from having one agent exert first-best effort *after* one of them retires outweighs the loss from the two agents exerting less effort *before* one of them retires (relative to the case in which they always work together) only if the project is sufficiently small. Hence, this retirement scheme is preferable if and only if  $|q_0| < \Theta_R$ .

*Remark 2.* This result implies that an asymmetric scheme that rewards the agents upon reaching (different) intermediate milestones can do better than the best symmetric one if the project is sufficiently small. Observe that the retirement scheme proposed above can be implemented using the following (asymmetric) *rewards-for-milestones* scheme. Let  $Q_1 = R$  (where  $|R| \leq \min\{|q_0|, T_{1,2}\}$ ), and suppose that agent 1 receives  $V$  as soon as the project is completed, while he receives no intermediate rewards. On the other hand, agent 2 receives the expected discounted value of  $B - V$  upon hitting  $R$  for the first time (i.e.,  $(B - V) \mathbb{E}_\tau [e^{-r\tau} | R]$ ), and he receives no further compensation, so that he effectively retires at that point. From Proposition 6 we know that there exists a *budget split*  $V$  and a threshold  $\Theta_R$  such that this scheme is preferable if  $|q_0| < \Theta_R$ .

We know that an asymmetric compensation scheme may be beneficial, because it enables the manager to dynamically decrease the team size as the project gets close to completion. In this

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<sup>27</sup>It is shown that (i) such a pair  $\{V_1, V_2\}$  exists, and (ii) the agent who will retire at  $R$  (say agent 2) receives a smaller payment than the other agent; i.e.,  $V_2 < V_1$ .

case, the asymmetry arises from the fact that agents are compensated upon reaching different milestones. The following result shows that an asymmetric scheme may be preferable even if the manager compensates the (identical) agents upon reaching the same milestone; namely, upon completing the project.

**Proposition 7.** *Suppose the manager has a fixed budget  $B > 0$ , and she employs two identical agents with quadratic effort costs whom she compensates upon completion of the project. Then for all  $\epsilon \in (0, \frac{B}{2}]$ , there exists a threshold  $T_\epsilon$  such that the manager is better off compensating the two agents asymmetrically such that  $V_1 = \frac{B}{2} + \epsilon$  and  $V_2 = \frac{B}{2} - \epsilon$  instead of symmetrically, if and only if the size of the project  $|q_0| \leq T_\epsilon$ .*

To see the intuition behind this result, note that  $\epsilon = \frac{B}{2}$  is equivalent to the case in which the manager employs a single agent, and from Proposition 5 we know that there exists a threshold  $T_{1,2}$  such that the manager is better off employing one agent instead of two if and only if  $|q_0| \leq T_{1,2}$ . The intermediate cases in which  $\epsilon \in (0, \frac{B}{2})$  can be thought of as if the manager employs a *full-time agent* and a *part-time* one. Part of the proof involves showing that the aggregate effort under an asymmetric scheme is larger compared to a symmetric one if and only if the project is sufficiently close to completion. Intuitively, this is because the full-time agent cannot free-ride on the other agent as much. By noting that the manager's objective is to allocate her budget so as to maximize the agents' expected aggregate effort along the evolution path of the project, it follows that this is best done by allocating it asymmetrically between the agents if the project is sufficiently small.

## 6 Concluding Remarks

This paper studies the dynamic collaboration of a team on a project that gradually progresses towards completion. The main result is that members of a larger team work harder than those of a smaller team if and only if the project is sufficiently far from completion. On the other hand, when the project is close to completion, free-riding becomes so severe that a larger team may on aggregate exert less effort than a smaller team.

The primary contribution of this paper is that it provides a rationale for the formation of project teams even without complementary skills, mutual monitoring and peer pressure among the team members, or non-pecuniary benefits from teamwork. Moreover, when contrasted to the results of Bonatti and Hörner (2011), it explains why the adoption of teamwork is more prevalent in manufacturing (where tasks are predominantly evolutionary) than in service operations (where tasks have more of a breakthrough feature).

This paper opens several opportunities for future research. First, Georgiadis, Lippman and Tang (2012) consider the case in which the project size is endogenous. Motivated by projects that involve *design* or *quality* objectives which are often difficult to define in advance, they examine how the manager's optimal project size depends on her ability to commit to a given project size in advance. Second, this paper provides several testable predictions that lend themselves to empirical or experimental investigation; in particular, Ederer, Georgiadis and Nunnari test the predictions of Theorem 2 using laboratory experiments. Third, one may incorporate time into the agents' contracts so that each agent's compensation also depends on the completion time of the project. Fourth, incorporating learning into the problem might be a fruitful avenue for future research: each agent might gradually learn about the difficulty of the project, his ability to carry it out, or about the other agents' valuation of the project. Fifth, optimal contracting for incentivizing a group of agents to complete a project is a problem that deserves further exploration; for example, using an approach in the mold of Sannikov (2008). Sixth, one may consider the case in which the state of the project can only be observed imperfectly, in which case the agents would need to update their beliefs about how close the project is to completion over time, and base their strategies on those beliefs. Finally, from an applied perspective, it might be interesting to examine how a project can be split into subprojects.

# A Additional Results

## A.1 Equilibria with Non-Markovian Strategies

I have assumed that agents' strategies are Markovian so that at every moment, each agent's effort is a function of only the current state of the project  $q_t$ . This raises the question whether agents can increase their expected discounted payoff by adopting non-Markovian strategies, so that their effort depends on the entire evolution path of the project  $\{q_s\}_{s \leq t}$ . Sannikov and Skrzypacz (2007) study a related model in which agents can change their actions only at times  $t = 0, \Delta, 2\Delta, \dots$ , where  $\Delta > 0$  (but *small*), and the information structure is similar; i.e., the state variable evolves according to a diffusion process whose drift is influenced by the agents' actions. They show that the payoffs from the best symmetric Public Perfect equilibrium (hereafter PPE) converge to the payoffs corresponding to the MPE as  $\Delta \rightarrow 0$  (see their Proposition 5).

A natural discrete-time analog of the model considered in this paper is one in which at  $t \in \{0, \Delta, 2\Delta, \dots\}$  each agent chooses his effort level  $a_{i,t}$  at cost  $c(a_{i,t}) \Delta$ , and at  $t + \Delta$  the state of the project is equal to  $q_{t+\Delta} = q_t + (\sum_{i=1}^n a_{i,t}) \Delta + \epsilon_{t+\Delta}$ , where  $\epsilon_{t+\Delta} \sim N(0, \sigma^2 \Delta)$ . In light of the similarities between this model and the model in Section VI of Sannikov and Skrzypacz (2007), it is reasonable to conjecture that in the continuous-time game, there does not exist a PPE in which agents can achieve a higher expected discounted payoff than the MPE at any state of the project. However, because a rigorous proof is difficult for the continuous-time game and the focus of this paper is on team formation, a formal analysis of non-Markovian PPE of this game is left for future work.

Nevertheless, it is useful to present some intuition. Following Abreu, Pearce and Stacchetti (1986), an optimal PPE involves a collusive regime and a punishment regime, and in every period, the decision whether to remain in the collusive regime or to switch is guided by the outcome in that period alone. In the context of this model, at  $t + \Delta$ , each agent will base his decision on  $\frac{q_{t+\Delta} - q_t}{\Delta}$ . As  $\Delta$  decreases, two forces influence the scope of cooperation. First, the gain from a deviation in a single period decreases, which helps cooperation. On the other hand, because  $\mathbb{V}\left(\frac{q_{t+\Delta} - q_t}{\Delta}\right) = \frac{\sigma^2}{\Delta}$ , the agents must decide whether to switch to the punishment regime by observing noisier information, which increases the probability of type I errors (i.e., triggering a punishment when no deviation has occurred), thus hurting cooperation. As Sannikov and Skrzypacz (2007) show, the latter force becomes overwhelmingly stronger than the former as  $\Delta \rightarrow 0$ , thus eradicating any gains from cooperation.

## A.2 Linear Effort costs

The assumption that effort costs are convex affords tractability as it allows for comparative statics despite the fact that the underlying system of HJB equations does not have a closed-form solution. However, convex effort costs also favor larger teams. Therefore, it is useful to examine how the comparative statics with respect to the team size extend to the case in which effort costs are linear; i.e.,  $c(a) = a$ . In this case, the marginal value of effort is equal to  $J'_i(q) - 1$ , so that agents find it optimal to exert the largest possible effort level if  $J'_i(q) \geq 1$ , and the smallest possible effort level otherwise. As a result, it is necessary to impose bounds on the minimum and maximum effort that each agent can exert at any moment. Let us assume that  $a \in [0, 1]$ . Moreover, suppose that agents are symmetric, and  $\sigma = 0$  so that the project evolves deterministically.<sup>28</sup> By using (2) subject to (3) and the corresponding first order condition, it follows that a unique *project-completing* MPE exists if  $q \geq \psi_n$ , where  $\psi_n = \frac{n}{r} \ln \left( \frac{n}{rV_n+1} \right)$ , it is symmetric, and each agent's discounted payoff and effort strategy satisfies

$$J_n(q) = \left[ -\frac{1}{r} + \left( V_n + \frac{1}{r} \right) e^{\frac{rq}{n}} \right] \mathbf{1}_{\{q \geq \psi_n\}} \text{ and } a_n(q) = \mathbf{1}_{\{q \geq \psi_n\}},$$

respectively.<sup>29</sup> Observe that agents have stronger incentives the closer the project is to completion, as evidenced by the facts that  $J''_n(q) \geq 0$  for all  $q$ , and  $a_n(q) = 1$  if and only if  $q \geq \psi_n$ . To investigate how the agents' incentives depend on the team size, one needs to examine how  $\psi_n$  depends on  $n$ . This threshold decreases in the team size  $n$  under both allocation schemes (i.e., both if  $V_n = V$  and  $V_n = \frac{V}{n}$  for some  $V > 0$ ) if and only if  $n$  is sufficiently small. This implies that members of an  $(n + 1)$  – member team have stronger incentives relative to those of an  $n$  – member team as long as  $n$  is sufficiently small.

If agents maximize the team's rather than their individual discounted payoff, then the first-best threshold  $\hat{\psi}_n = \frac{n}{r} \ln \left( \frac{1}{rV+1} \right)$ , and it is straightforward to show that it decreases in  $n$  under both allocation schemes. Therefore, similar to the case in which effort costs are convex, members of a larger team always have stronger incentives than those of a smaller one.

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<sup>28</sup>While the corresponding HJB equation can be solved analytically if effort costs are linear, the solution is too complex to obtain the desired comparative statics if  $\sigma > 0$ .

<sup>29</sup>Note that there exists another equilibrium in which no agent exerts any effort and the project is never completed if  $q \in \left[ \frac{n}{r} \ln \left( \frac{n}{rV+1} \right), \frac{1}{r} \ln \left( \frac{1}{rV+1} \right) \right)$ .

## B Proofs

*Proof of Theorem 1.* This proof is organized in 7 parts. I first show that a MPE for the game defined by (1) exists. Next I show that properties (i) thru (iii) hold, and that the value functions are infinitely differentiable. Then I show that with symmetric agents, the equilibrium is also symmetric. Finally, I show that the solution to the system of boundary value ODE is unique when the game comprises of  $n$  symmetric, or 2 asymmetric agents.

### Part I: Existence of a MPE.

To show that a MPE exists, it suffices to show that a solution satisfying the system of ordinary nonlinear differential equations defined by (4) subject to the boundary conditions (3) for all  $i = 1, \dots, n$  exists.

To begin, fix some arbitrary  $N \in \mathbb{N}$  and rewrite (4) and (3) as

$$J''_{i,N}(q) = \frac{2}{\sigma^2} \left[ rJ_{i,N}(q) + c(f(J'_{i,N}(q))) - \left( \sum_{j=1}^n f(J'_{j,N}(q)) \right) J'_{i,N}(q) \right] \quad (7)$$

subject to  $J_{i,N}(-N) = 0$  and  $J_{i,N}(0) = V_i$

for all  $i$ . Let  $g_i(J_N, J'_N)$  denote the the RHS of (7), where  $J_N$  and  $J'_N$  are vectors whose  $i^{\text{th}}$  row corresponds to  $J_{i,N}(q)$  and  $J'_{i,N}(q)$ , respectively, and note that  $g_i(\cdot, \cdot)$  is continuous. Now fix some arbitrary  $K > 0$ , and define a new function

$$g_{i,K}(J_N, J'_N) = \max \{ \min \{ g_i(J_N, J'_N), K \}, -K \}.$$

Note that  $g_{i,K}(\cdot, \cdot)$  is continuous and bounded. Therefore, by Lemma 4 in Hartman (1960), there exists a solution to  $J''_{i,N,K} = g_{i,K}(J_{i,N,K}, J'_{i,N,K})$  subject to  $J_{i,N,K}(-N) = 0$  and  $J_{i,N,K}(0) = V_i$  for all  $i$ . This Lemma, which is due to Scorza-Dragnoni (1935), states:

Let  $g(q, J, J')$  be a continuous and bounded (vector-valued) function for  $\alpha \leq q \leq \beta$  and arbitrary  $(J, J')$ . Then, for arbitrary  $q_\alpha$  and  $q_\beta$ , the system of differential equations  $J'' = g(q, J, J')$  has at least one solution  $J = J(q)$  satisfying  $J(\alpha) = q_\alpha$  and  $J(\beta) = q_\beta$ .

The next part of the proof involves showing that there exists a  $\bar{K}$  such that  $g_{i,K}(J_{i,N,K}(q), J'_{i,N,K}(q)) \in (-\bar{K}, \bar{K})$  for all  $i, K$  and  $q$ , which will imply that the solution  $J_{i,N,\bar{K}}(\cdot)$  satisfies (7) for all  $i$ . The final step involves showing that a solution exists when  $N \rightarrow \infty$ , so that the first boundary condition in (7) is replaced by  $\lim_{q \rightarrow -\infty} J_i(q) = 0$ .

First, I show that  $0 \leq J_{i,N,K}(q) \leq V_i$  and  $J'_{i,N,K}(q) \geq 0$  for all  $i$  and  $q$ . Because  $J_{i,N,K}(0) > J_{i,N,K}(-N) = 0$ , either  $J_{i,N,K}(\cdot)$  is increasing, or it has an interior global extreme point. If the former is true, then the desired inequality holds. Suppose the latter is true and let  $z^*$  denote such interior global extreme point. By noting that  $J_{i,N,K}(\cdot)$  is at least twice differentiable,  $J'_{i,N,K}(z^*) = 0$ , and hence  $J''_{i,N,K}(z^*) = \max\{\min\{\frac{2r}{\sigma^2}J_{i,N,K}(z^*), K\}, -K\}$ . Suppose  $z^*$  is a global maximum. Then  $J''_{i,N,K}(z^*) \leq 0 \implies J_{i,N,K}(z^*) \leq 0$ , which contradicts the fact that  $J_{i,N,K}(0) > 0$ . Now suppose that  $z^*$  is a global minimum. Then  $J''_{i,N,K}(z^*) \geq 0 \implies J_{i,N,K}(z^*) \geq 0$ . Therefore either  $J_{i,N,K}(\cdot)$  is increasing, or it has an interior global minimum  $z^*$  such that  $J_{i,N,K}(z^*) \geq 0$ . As a result,  $0 \leq J_{i,N,K}(q) \leq V_i$  for all  $i$  and  $q$ .

Next, let us focus on  $J'_{i,N,K}(\cdot)$ . Suppose that there exists a  $z^{**}$  such that  $J'_{i,N,K}(z^{**}) < 0$ . Because  $J_{i,N,K}(-N) = 0$ , either  $J_{i,N,K}(\cdot)$  is decreasing on  $[-N, z^{**}]$ , or it has a local maximum  $\bar{z} \in (-N, z^{**})$ . If the former is true, then  $J'_{i,N,K}(z^{**}) < 0$  implies that  $J_{i,N,K}(q) < 0$  for some  $q \in (-N, z^{**})$ , which is a contradiction because  $J_{i,N,K}(q) \geq 0$  for all  $q$ . So the latter must be true. Then  $J'_{i,N,K}(\bar{z}) = 0$  implies that  $J''_{i,N,K}(\bar{z}) = \max\{\min\{\frac{2r}{\sigma^2}J_{i,N,K}(\bar{z}), K\}, -K\}$ . However, because  $\bar{z}$  is a maximum,  $J''_{i,N,K}(\bar{z}) \leq 0$ , and together with the fact that  $J_{i,N,K}(q) \geq 0$  for all  $q$ , this implies that  $J_{i,N,K}(q) = 0$  for all  $q \in [-N, z^{**}]$ . But since  $J'_{i,N,K}(z^{**}) < 0$ , it follows that  $J_{i,N,K}(q) < 0$  for some  $q$  in the neighborhood of  $z^{**}$ , which is a contradiction. Therefore, it must be the case that  $J'_{i,N,K}(q) \geq 0$  for all  $i$  and  $q$ .

The next step involves establishing that there exists an  $\bar{A}$ , independent of  $N$  and  $K$ , such that  $J'_{i,N,K}(q) < \bar{A}$  for all  $i$  and  $q$ . First, let  $S_{N,K}(q) = \sum_{i=1}^n J_{i,N,K}(q)$ . By summing  $J''_{i,N,K} = g_{i,K}(J_{i,N,K}, J'_{i,N,K})$  over  $i$ , using that (i)  $0 \leq J_{i,N,K}(q) \leq V_i$  and  $0 \leq J'_{i,N,K}(q) \leq S'_{N,K}(q)$  for all  $i$  and  $q$ , (ii)  $f(x) = x^{1/p}$ , and (iii)  $c(x) \leq x c'(x)$  for all  $x \geq 0$ , and letting  $\Gamma = r \sum_{i=1}^n V_i$ , we have that for all  $q$

$$\begin{aligned} |S''_{N,K}(q)| &\leq \frac{2}{\sigma^2} \sum_{i=1}^n \left[ r J_{i,N,K}(q) + c(f(J'_{i,N,K}(q))) + \left[ \sum_{j=1}^n f(J'_{j,N,K}(q)) \right] J'_{i,N,K}(q) \right] \\ &\leq \frac{2}{\sigma^2} \left[ \Gamma + \sum_{i=1}^n c'(c^{-1}(J'_{i,N,K}(q))) c^{-1}(J'_{i,N,K}(q)) + S'_{N,K}(q) \sum_{j=1}^n f(J'_{j,N,K}(q)) \right] \\ &\leq \frac{4}{\sigma^2} \left[ \Gamma + n S'_{N,K}(q) f(S'_{N,K}(q)) \right] = \frac{4}{\sigma^2} \left[ \Gamma + n (S'_{N,K}(q))^{\frac{p+1}{p}} \right]. \end{aligned}$$

By noting that  $S_{N,K}(0) = \sum_{i=1}^n V_i$ ,  $S_{N,K}(-N) = 0$ , and applying the mean value theorem, it follows that there exists a  $z^* \in [-N, 0]$  such that  $S'_{N,K}(z^*) = \frac{\sum_{i=1}^n V_i}{N}$ . It follows that for

all  $z \in [-N, 0]$

$$\sum_{i=1}^n V_i > \int_{z^*}^z S'_{N,K}(q) dq \geq \frac{\sigma^2}{4} \int_{z^*}^z S'_{N,K}(q) \frac{S''_{N,K}(q)}{\Gamma + n (S'_{N,K}(q))^{\frac{p+1}{p}}} dq \geq \frac{\sigma^2}{4} \int_0^{S'_N(z)} \frac{s}{\Gamma + ns^{\frac{p+1}{p}}} ds,$$

where I let  $s = S'_{N,K}(q)$  and used that  $S'_{N,K}(q) S''_{N,K}(q) dq = S'_{N,K}(q) dS'_{N,K}(q)$ . It suffices to show that there exists a  $\bar{A} < \infty$  such that  $\frac{\sigma^2}{4} \int_0^{\bar{A}} \frac{s}{\Gamma + ns^{\frac{p+1}{p}}} ds = \sum_{i=1}^n V_i$ . This will imply that  $S'_{N,K}(q) < \bar{A}$ , and consequently  $J'_{i,N,K}(q) \leq \bar{A}$  for all  $q \in [-N, 0]$ . To show that such  $\bar{A}$  exists, it suffices to show that  $\int_0^\infty \frac{s}{\Gamma + ns^{\frac{p+1}{p}}} ds = \infty$ . First, observe that if  $p = 1$ , then  $\int_0^\infty \frac{s}{\Gamma + ns^2} ds = \frac{1}{2n} \ln(\Gamma + ns^2) \Big|_0^\infty = \infty$ . By noting that  $\frac{s}{\Gamma + ns^2}$  is bounded for all  $s \in [0, 1]$ ,  $\frac{s}{\Gamma + ns^{\frac{p+1}{p}}} > \frac{s}{\Gamma + ns^2}$  for all  $s > 1$  and  $p > 1$ , and  $\int_0^\infty \frac{s}{\Gamma + ns^2} ds = \infty$ , integrating both sides over  $[0, \infty]$  yields the desired inequality.

Because  $\bar{A}$  is independent of both  $N$  and  $K$ , this implies that  $J'_{i,N,K}(q) \in [0, \bar{A}]$  for all  $q \in [-N, 0]$ ,  $N \in \mathbb{N}$  and  $K > 0$ . In addition, we know that  $J_{i,N,K}(q) \in [0, V_i]$  for all  $q \in [-N, 0]$ ,  $N \in \mathbb{N}$  and  $K > 0$ . Now let  $\bar{K} = \max_i \left\{ \frac{2}{\sigma^2} [rV_i + c(f(\bar{A}))] \right\}$ , and observe that a solution to  $J''_{i,N,\bar{K}} = g_{i,\bar{K}}(J_{N,\bar{K}}, J'_{N,\bar{K}})$  subject to  $J_{i,N,\bar{K}}(-N) = 0$  and  $J_{i,N,\bar{K}}(0) = V_i$  for all  $i$  exists, and  $g_{i,\bar{K}}(J_{N,\bar{K}}(q), J'_{N,\bar{K}}(q)) = g(J_{i,N,\bar{K}}(q), J'_{N,\bar{K}}(q))$  for all  $i$  and  $q \in [-N, 0]$ . Therefore,  $J_{i,N,\bar{K}}(\cdot)$  solves (7) for all  $i$ .

To show that a solution for (7) exists at the limit as  $N \rightarrow \infty$ , I use the Arzela-Ascoli theorem, which states:

Consider a sequence of real-valued continuous functions  $(f_n)_{n \in \mathbb{N}}$  defined on a closed and bounded interval  $[a, b]$  of the real line. If this sequence is uniformly bounded and equicontinuous, then there exists a subsequence  $(f_{n_k})$  that converges uniformly.

Recall that  $0 \leq J_{i,N}(q) \leq V_i$  and that there exists a constant  $\bar{A}$  such that  $0 \leq J'_{i,N}(q) \leq \bar{A}$  on  $[-N, 0]$  for all  $i$  and  $N > 0$ . Hence the sequences  $\{J_{i,N}(\cdot)\}$  and  $\{J'_{i,N}(\cdot)\}$  are uniformly bounded and equicontinuous on  $[-N, 0]$ . By applying the Arzela-Ascoli theorem to a sequence of intervals  $[-N, 0]$  and letting  $N \rightarrow \infty$ , it follows that the system of ODE defined by (4) has at least one solution satisfying the boundary conditions (3) for all  $i$ .

**Part II:**  $J_i(q) > 0$  for all  $q$  and  $i$ .

By the boundary conditions we have that  $\lim_{q \rightarrow -\infty} J_i(q) = 0$  and  $J_i(0) = V_i > 0$ . Suppose that there exists an interior  $z^*$  that minimizes  $J_i(\cdot)$  on  $(-\infty, 0]$ . Clearly  $z^* < 0$ . Then

$J'_i(z^*) = 0$  and  $J''_i(z^*) \geq 0$ , which by applying (4) imply that

$$rJ_i(z^*) = \frac{\sigma^2}{2} J''_i(z^*) \geq 0.$$

Because  $\lim_{q \rightarrow -\infty} J_i(q) = 0$ , it follows that  $J_i(z^*) = 0$ . Next, let  $z^{**} = \arg \max_{q \leq z^*} \{J_i(q)\}$ . If  $z^{**}$  is on the boundary of the desired domain, then  $J_i(q) = 0$  for all  $q \leq z^*$ . Suppose that  $z^{**}$  is interior. Then  $J'_i(z^{**}) = 0$  and  $J''_i(z^{**}) \leq 0$  imply that  $J_i(z^{**}) \leq 0$ , so that  $J_i(q) = J'_i(q) = 0$  for all  $q < z^*$ .

Using (4) we have that

$$|J''_i(q)| \leq \frac{2r}{\sigma^2} |J_i(q)| + \frac{2}{\sigma^2} (n+1) f(\bar{A}) |J'_i(q)|,$$

where this bound follows from part I of the proof. Now let  $h_i(q) = |J_i(q)| + |J'_i(q)|$ , and observe that  $h_i(q) = 0$  for all  $q < z^*$ ,  $h_i(q) \geq 0$  for all  $q$ , and

$$h'_i(q) \leq |J'_i(q)| + |J''_i(q)| \leq \frac{2r}{\sigma^2} |J_i(q)| + \frac{2}{\sigma^2} \left[ f_i(\bar{A}) + \sum_{j=1}^n f_j(\bar{A}) + \frac{\sigma^2}{2} \right] |J'_i(q)| \leq C h_i(q),$$

where  $C = \frac{2}{\sigma^2} \max \left\{ r, (n+1) f(\bar{A}) + \frac{\sigma^2}{2} \right\}$ . Fix some  $\hat{z} < z^*$ , and applying the differential form of Grönwall's inequality yields  $h_i(q) \leq h_i(\hat{z}) \exp \left( \int_{\hat{z}}^q C dx \right)$  for all  $q$ . Because (i)  $h_i(\hat{z}) = 0$ , (ii)  $\exp \left( \int_{\hat{z}}^q C dx \right) < \infty$  for all  $q$ , and (iii)  $h_i(q) \geq 0$  for all  $q$ , this inequality implies that  $J_i(q) = 0$  for all  $q$ . However this contradicts the fact that  $J_i(0) = V_i > 0$ . As a result,  $J_i(\cdot)$  cannot have an interior minimum, and there cannot exist a  $z^* > -\infty$  such that  $J_i(q) = 0$  for all  $q \leq z^*$ . Hence  $J_i(q) > 0$  for all  $q$ .

**Part III:**  $J'_i(q) > 0$  for all  $q$  and  $i$ .

Pick a  $K$  such that  $J_i(0) < J_i(K) < V_i$ . Such  $K$  is guaranteed to exist, because  $J_i(\cdot)$  is continuous and  $J_i(0) > 0 = \lim_{q \rightarrow -\infty} J_i(q)$ . Then by the mean-value theorem there exists a  $z^* \in (K, 0)$  such that  $J'_i(z^*) = \frac{J_i(0) - J_i(K)}{-K} = \frac{V_i - J_i(K)}{-K} > 0$ . Suppose that there exists a  $z^{**} \leq 0$  such that  $J'_i(z^{**}) \leq 0$ . Then by the intermediate value theorem, there exists a  $\bar{z}$  between  $z^*$  and  $z^{**}$  such that  $J'_i(\bar{z}) = 0$ , which using (4) and part II implies that  $rJ_i(\bar{z}) = \frac{\sigma^2}{2} J''_i(\bar{z}) > 0$  (i.e.,  $\bar{z}$  is a local minimum). Consider the interval  $(-\infty, \bar{z}]$ . Because  $\lim_{q \rightarrow -\infty} J_i(q) = 0$ ,  $J_i(\bar{z}) > 0$  and  $J''_i(\bar{z}) > 0$ , there exists an interior local maximum  $\hat{z} < \bar{z}$ . Since  $\hat{z}$  is interior, it must be the case that  $J'_i(\hat{z}) = 0$  and  $J''_i(\hat{z}) \leq 0$ , which using (4) implies that  $J_i(\hat{z}) \leq 0$ . However this contradicts the fact that  $J_i(q) > 0$  for all  $q$ . As a result there cannot exist a  $\bar{z} \leq 0$  such that  $J'_i(\bar{z}) \leq 0$ . Together with part II, this proves properties (i) and (ii).

**Part IV:**  $J_i(q)$  is infinitely differentiable on  $(-\infty, 0]$  for all  $i$ .

By noting that  $\lim_{q \rightarrow -\infty} J_i(q) = \lim_{q \rightarrow -\infty} J'_i(q) = 0$  for all  $i$ , and by twice integrating both sides of (7) over the interval  $(-\infty, q]$ , we have that

$$J_i(q) = \int_{-\infty}^q \int_{-\infty}^y \frac{2r}{\sigma^2} J_i(z) + \frac{2}{\sigma^2} \left[ c(f(J'_i(z))) - \left( \sum_{j=1}^n f(J'_j(z)) \right) J'_i(z) \right] dz dy.$$

Recall that  $c(a) = \frac{a^{p+1}}{p+1}$ ,  $f(x) = x^{1/p}$ , and  $J'_i(q) > 0$  for all  $q$ . Since  $J_i(q)$  and  $J'_i(q)$  satisfy (4) subject to the boundary conditions (3) for all  $i$ ,  $J_i(q)$  and  $J'_i(q)$  are continuous for all  $i$ . As a result, the function under the integral is continuous and infinitely differentiable in  $J_i(z)$  and  $J'_i(z)$  for all  $i$ . Because  $J_i(q)$  is differentiable twice more than the function under the integral, the desired result follows by induction.

**Part V:**  $J''_i(q) > 0$  and  $a'_i(q) > 0$  for all  $q$  and  $i$ .

I have thus far established that for all  $q$ ,  $J_i(q) > 0$  and  $J'_i(q) > 0$ . By applying the envelope theorem to (4) we have that

$$rJ'_i(q) = [f(J'_i(q)) + A_{-i}(q)] J''_i(q) + \frac{\sigma^2}{2} J'''_i(q), \quad (8)$$

where  $A_{-i}(q) = \sum_{j \neq i}^n f(J'_j(q))$ . Choose some finite  $z^* \leq 0$ , and let  $z^{**} = \arg \max \{J'_i(q) : q \leq z^*\}$ . By part III,  $J'_i(z^{**}) > 0$ . Because  $\lim_{q \rightarrow -\infty} J'_i(q) = 0$ , either  $z^{**} = z^*$ , or  $z^{**}$  is interior. Suppose  $z^{**}$  is interior. Then  $J''_i(z^{**}) = 0$  and  $J'''_i(z^{**}) \leq 0$ , which using (8) implies that  $J'_i(z^{**}) \leq 0$ . However this contradicts the fact that  $J'_i(z^{**}) > 0$ , and therefore  $J'_i(\cdot)$  does not have an interior maximum on  $(-\infty, z^*]$  for any  $z^* \leq 0$ . Therefore  $z^{**} = z^*$ , and hence  $J'_i(\cdot)$  is strictly increasing; i.e.,  $J'_i(q) > 0$  for all  $q$ . By differentiating  $a_i(q)$  we have that

$$\frac{d}{dq} a_i(q) = \frac{d}{dq} c'^{-1}(J'_i(q)) = \frac{J''_i(q)}{c''(c'^{-1}(J'_i(q)))} > 0.$$

**Part VI:** When the agents are symmetric, the MPE is also symmetric.

Suppose agents are symmetric; i.e., they have identical effort costs, patience levels, and they receive the same reward upon completing the project. In any MPE,  $\{J_i(\cdot)\}_{i=1}^n$  must satisfy (4) subject to (3). Arbitrarily pick two agents  $i$  and  $j$ , and let  $\Delta(q) = J_i(q) - J_j(q)$ . Observe that  $\Delta(\cdot)$  is smooth, and  $\lim_{q \rightarrow -\infty} \Delta(q) = \Delta(0) = 0$ . Therefore either  $\Delta(\cdot) \equiv 0$  on  $(-\infty, 0]$ , which implies that  $J_i(\cdot) \equiv J_j(\cdot)$  on  $(-\infty, 0]$  for all  $i \neq j$  and hence the equilibrium is symmetric, or  $\Delta(\cdot)$  has at least one interior global extreme point. Suppose the latter is true, and denote this extreme point by  $z^*$ . Then and by using (4) and the fact that  $\Delta'(z^*) = 0$ ,

we have  $r\Delta(z^*) = \frac{\sigma^2}{2}\Delta''(z^*)$ . Suppose that  $z^*$  is a maximum. Then  $\Delta''(z^*) \leq 0$ , which implies that  $\Delta(z^*) \leq 0$ . However, because  $\Delta(0) = 0$  and  $z^*$  is assumed to be a maximum,  $\Delta(z^*) = 0$ . Next, suppose that  $z^*$  is a minimum. Then  $\Delta''(z^*) \geq 0$ , which implies that  $\Delta(z^*) \geq 0$ . However, because  $\Delta(0) = 0$  and  $z^*$  is assumed to be a minimum,  $\Delta(z^*) = 0$ . Therefore it must be the case that  $\Delta(\cdot) \equiv 0$  on  $(-\infty, 0]$ . Since  $i$  and  $j \neq i$  were chosen arbitrarily,  $J_i(\cdot) \equiv J_j(\cdot)$  on  $(-\infty, 0]$  for all  $i \neq j$ , which implies that the equilibrium is symmetric.

**Part VII:** *The system of ordinary nonlinear differential equations defined by (4) for (a)  $n \in \mathbb{N}$  symmetric agents, and (b) 2 asymmetric agents, has at most one solution satisfying the boundary conditions (3).*

CASE (A): I first prove uniqueness for  $n$  symmetric agents. From Part VI of the proof, we know that if agents are symmetric, then the MPE is symmetric. Therefore to facilitate exposition, I drop the notation for the  $i^{\text{th}}$  agent. Any solution  $J(\cdot)$  must satisfy

$$rJ(q) = -c(f(J'(q))) + nf(J'(q))J'(q) + \frac{\sigma^2}{2}J''(q) \text{ subject to } \lim_{q \rightarrow -\infty} J(q) = 0 \text{ and } J(0) = V.$$

Suppose that there exist 2 functions  $J_A(q)$ ,  $J_B(q)$  that satisfy the above boundary value problem. Then define  $D(q) = J_A(q) - J_B(q)$ , and note that  $D(\cdot)$  is smooth and  $\lim_{q \rightarrow -\infty} D(q) = D(0) = 0$ . Hence either  $D(\cdot) \equiv 0$  in which case the proof for (a) is complete, or  $D(\cdot)$  has an interior global extreme point  $z^*$ . Suppose the latter is true. Then  $D'(z^*) = 0$ , which implies that  $rD(z^*) = \frac{\sigma^2}{2}D''(z^*)$ . Suppose that  $z^*$  is a global maximum. Then  $D''(z^*) \leq 0$  implies that  $D(z^*) \leq 0$ , and  $D(0) = 0$  implies that  $D(z^*) = 0$  and  $D(q) \leq 0$  for all  $q$ . Hence either  $D(\cdot) \equiv 0$  or  $z^*$  is a global minimum. Suppose the latter is true. Then  $D''(z^*) \geq 0$  implies that  $D(z^*) \geq 0$ , and  $D(0) = 0$  implies that  $D(z^*) = 0$  and  $D(q) \geq 0$  for all  $q$ . Therefore it must be the case that  $D(\cdot) \equiv 0$  and the proof for (a) is complete.

CASE (B): Now consider (4) for the case with 2 asymmetric agents. Any solution  $J_1(q)$  and  $J_2(q)$  must satisfy

$$\begin{aligned} J_1''(q) &= -\frac{2}{\sigma^2} \frac{p}{p+1} [J_1'(q)]^{\frac{p+1}{p}} - \frac{2}{\sigma^2} J_1'(q) [J_2'(q)]^{\frac{1}{p}} + \frac{2r}{\sigma^2} J_1(q) \quad \text{and} \\ J_2''(q) &= -\frac{2}{\sigma^2} \frac{p}{p+1} [J_2'(q)]^{\frac{p+1}{p}} - \frac{2}{\sigma^2} [J_1'(q)]^{\frac{1}{p}} J_2'(q) + \frac{2r}{\sigma^2} J_2 \end{aligned}$$

subject to  $J_i(0) = V_i$  and  $\lim_{q \rightarrow -\infty} J_i(q) = 0$  for all  $i = \{1, 2\}$ .

To show that there exists a unique solution to the above system of ODE, I shall use Theorem

5 in Hartman (1960), which states

Let  $g(q, J, J')$  be defined on  $D(T, R)$  (i.e., be a continuous vector value function on  $[-N, 0]$  such that  $|J_i(q)| \leq V_i$  for all  $q$  and  $i$ ) and possess continuous partial derivatives with respect to the components of  $J$  and  $J'$ . Let  $F(q, J, J')$  and  $G(q, J, J')$  denote the Jacobian matrices

$$F(q, J, J') = \frac{\partial g}{\partial J} \quad \text{and} \quad G(q, J, J') = \frac{\partial g}{\partial J'}$$

and suppose that  $4F - GG^T \succeq 0$ . Then (7) has at most one solution satisfying  $J_i(0) = V_i$  and  $J_i(-N) = 0$  for all  $i \in \{1, 2\}$ .

Clearly in this case  $g(q, J, J')$  is continuous,  $|J_i(q)|$  is bounded for all  $q$  and  $i$ , and possesses continuous partial derivatives with respect to  $J_i$  and  $J'_i$  for all  $i$ . Therefore it suffices to show that  $4F - GG^T \succeq 0$  holds for all  $N > 0$  and by letting  $N \rightarrow \infty$  conclude that the above system of 2 ODE has a unique solution on  $(-\infty, 0]$ . We have

$$F(q) = \frac{2}{\sigma^2} \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix} \quad \text{and} \quad G(q) = -\frac{2}{\sigma^2} \begin{bmatrix} [J'_1(q)]^{1/p} + [J'_2(q)]^{1/p} & \frac{1}{p} \frac{J'_1(q)}{J'_2(q)} [J'_2(q)]^{1/p} \\ \frac{1}{p} \frac{J'_2(q)}{J'_1(q)} [J'_1(q)]^{1/p} & [J'_1(q)]^{1/p} + [J'_2(q)]^{1/p} \end{bmatrix}.$$

To facilitate exposition, let us denote (only for this proof)  $\alpha = [J'_1(q)]^{1/p}$ ,  $\beta = [J'_2(q)]^{1/p}$  and  $\kappa = \frac{J'_1(q)}{J'_2(q)}$ . By noting that  $J'_i(q) > 0$  for all  $q$  and  $i$ , it follows that  $0 < \kappa < \infty$ . Then

$$4F - GG^T = \frac{2}{\sigma^2} \begin{bmatrix} 4r + (\alpha + \beta)^2 + \left(\frac{\kappa\beta}{p}\right)^2 & (\alpha + \beta) \left(\frac{\alpha}{\kappa p} + \frac{\kappa\beta}{p}\right) \\ (\alpha + \beta) \left(\frac{\alpha}{\kappa p} + \frac{\kappa\beta}{p}\right) & 4r + (\alpha + \beta)^2 + \left(\frac{\alpha}{\kappa p}\right)^2 \end{bmatrix}.$$

To check that the above matrix is positive definite I use Sylvester's criterion. First note that  $4r_1 + (\alpha + \beta)^2 + \left(\frac{\kappa\beta}{p}\right)^2 > 0$ . The determinant of the above matrix is equal to

$$16r^2 + 8(\alpha + \beta)^2 r + 4r \left[ \left(\frac{\alpha}{\kappa p}\right)^2 + \left(\frac{\kappa\beta}{p}\right)^2 \right] + (\alpha + \beta)^2 \left[ (\alpha + \beta)^2 - \frac{\alpha\beta}{p^2} \right].$$

Clearly the first three terms are positive. By noting that  $p \geq 1$  and using the property that  $J'_i(q) > 0$  for all  $q$ , it follows that the last term is also positive. Hence the determinant of the above matrix is positive and uniqueness follows by applying Theorem 5 from Hartman (1960) for any given  $N > 0$  and letting  $N \rightarrow \infty$ .

In light of the fact that  $J'_i(q) > 0$  for all  $q$ , it follows that the first-order condition for

each agent's best response always binds. As a result, any MPE must satisfy the system of ODE defined by (4) subject to (3). Since this system of ODE has a unique solution with  $n$  symmetric or 2 asymmetric agents, it follows that in these two cases, the dynamic game defined by (1) has a unique MPE. □

*Proof of Proposition 1.* This proof is organized in 4 parts. To begin, let  $J_i(\cdot)$  denote the expected discounted payoff of each member of an  $n$ -person team with parameters  $\{r_i, c_i(\cdot), V_i\}$  who undertakes a project with volatility  $\sigma$ .

**Proof for property (i):** First, pick  $\alpha < 1$  and  $V$  such that  $V_1 = \alpha V_2 < V_2 = V$ , and let  $r = r_1 = r_2$ . Let  $D_V(q) = J_1(q) - J_2(q)$ , and note that it is smooth, and  $D_V(0) = (\alpha - 1)V < 0 = \lim_{q \rightarrow -\infty} D_V(q) = 0$ . Suppose that  $D_V(\cdot)$  has some interior extreme point, which I denote by  $z^*$ . Then  $D'_V(z^*) = 0$ , and by using (4) we have

$$rD_V(z^*) = \frac{\sigma^2}{2}D''_V(z^*).$$

Suppose that  $z^*$  is a global minimum. Then  $D''_V(z^*) \geq 0 \implies D_V(z^*) \geq 0$ , which contradicts the fact that  $D_V(0) < 0$ . So  $z^*$  must be a global maximum. Then  $D''_V(z^*) \leq 0 \implies D_V(z^*) \leq 0$ , which contradicts the fact that  $z^*$  is interior. Hence  $D_V(\cdot)$  cannot have any interior extreme points, and thus it must be decreasing for all  $q$ ; i.e.,  $D'_V(q) \leq 0$  for all  $q$  and  $D'_V(q) < 0$  for at least some  $q$ .

The next step involves showing that in fact,  $D'_V(q) < 0$  for all  $q$ . Suppose that there exists a  $z$  such that  $D'_V(z) = 0$ . If  $D_V(z) = 0$ , then  $\lim_{q \rightarrow -\infty} D_V(q) = 0$ , any interior maximum on  $(-\infty, z]$  must satisfy  $D_V(z) \leq 0$ , and any interior minimum must satisfy  $D_V(z) \geq 0$ . It follows that  $D_V(q) = D'_V(q) = 0$  for all  $q < z$ . So suppose that  $D_V(z) < 0$ , and let  $\hat{z} = \arg \min_{q \leq z} \{D_V(q)\}$ . Clearly,  $\hat{z} > -\infty$ . Second, to show that  $\hat{z} < z$ , suppose that the contrary is true; i.e.,  $\hat{z} = z$ . Then  $D'_V(z) = 0$ ,  $D_V(z) < 0$ , and (4) imply that  $D''_V(z) < 0$ , which contradicts the assumption that  $\hat{z}$  is a minimum. Hence  $\hat{z}$  is interior, so that  $D'_V(z) = 0$  and  $D''_V(z) \geq 0$ , which together with (4) imply that  $D_V(z) \geq 0$ . However, this contradicts the assumption that  $D_V(z) < 0$ . Therefore,  $D_V(z) = 0$ , and it follows that  $D_V(q) = D'_V(q) = 0$  for all  $q < z$ . Next, let  $M(q) = [J_1(q) - J_2(q)] + [J'_1(q) - J'_2(q)]$ , and note that  $M(q) \leq 0$  for all  $q$ ,  $M(0) < 0$ , and  $M(q) = 0$  for all  $q < z$ . By applying the differential form of Grönwall's inequality, it follows that  $M(q) = 0$  for all  $q$ , which contradicts the fact that  $M(0) < 0$ . Hence, I conclude that there does not exist a  $z$  such that  $D'_V(z) = 0$ . Therefore,  $D'_V(q) < 0$  for all  $q$ , which implies that  $a_1(q) < a_2(q)$  for all  $q$ .

**Proof for property (ii):** First pick  $\delta > 1$  and  $r$  such that  $r_1 = \delta r > r = r_2$ . Next, define  $D_r(q) = J_1(q) - J_2(q)$ . By noting that  $\lim_{q \rightarrow -\infty} D_r(q) = D_r(0) = 0$ , observe that either  $D_r(\cdot) \equiv 0$ , or  $D_r(\cdot)$  has at least one interior extreme point. Suppose  $D_r(\cdot) \equiv 0$ . Then  $D'_r(\cdot) \equiv D''_r(\cdot) \equiv 0$ , and using (4) we have that  $\delta J_1(\cdot) \equiv J_2(\cdot)$ . However this is a contradiction, because  $J_1(\cdot) \equiv J_2(\cdot)$ , and  $\delta > 1$ . Therefore  $D_r(\cdot)$  must have at least one interior extreme point, which I denote by  $z^*$ . By noting that  $D'_r(z^*) = 0$  and using (4), we have that

$$r [\delta J_1(z^*) - J_2(z^*)] = \frac{\sigma^2}{2} D''_r(z^*) .$$

Suppose that  $z^*$  is a global maximum. Then  $D''_r(z^*) \leq 0$ , and hence  $\delta J_1(z^*) - J_2(z^*) \leq 0$ . However because  $J_i(\cdot) > 0$  and  $\delta > 1$ , this implies that  $D_r(z^*) < 0 = D_r(0)$ , which contradicts the assumption that  $z^*$  is a global maximum. Therefore,  $z^*$  must be a global minimum, and  $D_r(q) \leq 0$  for all  $q$ .

I next show that  $D_r(\cdot)$  is single-troughed. Suppose it is not. Then I can find an interior local minimum  $z^*$  followed by an interior local maximum  $\bar{z} > z^*$ . Since  $\bar{z}$  is an interior maximum,  $D'_r(\bar{z}) = 0$  and  $D''_r(\bar{z}) \leq 0$ , and from (4) it follows that  $\delta J_1(\bar{z}) \leq J_2(\bar{z})$ . Because  $z^*$  is an interior minimum,  $D''_r(z^*) \geq 0$  implies that  $\delta J_1(z^*) \geq J_2(z^*) \Rightarrow -\delta J_1(z^*) \leq -J_2(z^*)$ , and by using  $\delta J_1(\bar{z}) \leq J_2(\bar{z})$ , we have that  $0 < \delta [J_1(\bar{z}) - J_1(z^*)] \leq J_2(\bar{z}) - J_2(z^*)$ , where the first inequality follows from Theorem 1 (iii) and the fact that  $\bar{z} > z^*$ . By assumption  $D_r(\bar{z}) > D_r(z^*)$ , which implies that  $J_2(\bar{z}) - J_2(z^*) < J_1(\bar{z}) - J_1(z^*)$ , so that

$$\delta [J_1(\bar{z}) - J_1(z^*)] \leq J_2(\bar{z}) - J_2(z^*) < J_1(\bar{z}) - J_1(z^*) ,$$

which contradicts the facts that  $\delta > 1$  and  $J_1(\bar{z}) - J_1(z^*) > 0$ . Hence  $D_r(\cdot)$  must be single-troughed. Because  $\lim_{q \rightarrow -\infty} D_r(q) = D_r(0) = 0$ , there exists a  $\Theta_r < 0$  such that  $D'_r(q) \leq 0$  if and only if  $q \leq \Theta_r$ . Finally, because  $c_1(\cdot) \equiv c_2(\cdot) \Rightarrow f_1(\cdot) \equiv f_2(\cdot)$ , it follows that  $a_1(q) \leq a_2(q)$  if and only if  $D'_r(q) \leq 0$ , or equivalently, if and only if  $q \leq \Theta_r$ .

**Proof for property (iii):** First pick  $\alpha > 1$  and  $\sigma$  such that  $\sigma_1^2 = \alpha \sigma_2^2 > \sigma_2^2 = \sigma^2$ . Let  $J_1(\cdot)$  and  $J_2(\cdot)$  denote each agent's expected discounted payoff associated with  $\sigma_1$  and  $\sigma_2$ , respectively. Moreover let  $D_\sigma(q) = J_1(q) - J_2(q)$  and observe that  $\lim_{q \rightarrow -\infty} D_\sigma(q) = D_\sigma(0) = 0$ . So either  $D_\sigma(\cdot) \equiv 0$  on  $(-\infty, 0]$ , or  $D_\sigma(\cdot)$  has some interior global extreme point. Suppose that  $D_\sigma(\cdot) \equiv 0$  on  $(-\infty, 0]$ . This implies that  $D_\sigma(q) = D'_\sigma(q) = D''_\sigma(q) = 0$

for all  $q$ , and using (4) it follows that for all  $q$

$$rD_\sigma(q) = \frac{\sigma^2}{2} [\alpha D_\sigma''(q) + (\alpha - 1) J_2''(q)] \implies J_2''(q) = 0.$$

However this contradicts Theorem 1 (iii), which implies that  $D_\sigma(\cdot)$  has at least one interior global extreme point, denoted by  $z^*$ . Then  $D'_\sigma(z^*) = 0$ , and using (4) yields  $rD_\sigma(z^*) = \frac{\sigma^2}{2} [\alpha D_\sigma''(z^*) + (\alpha - 1) J_2''(z^*)]$ . Suppose that  $z^*$  is a global minimum. Then  $D_\sigma''(z^*) \geq 0$ ,  $\alpha > 1$ , and  $J_2''(z^*) > 0$  imply that  $D_\sigma(z^*) > 0$ . However, this contradicts the fact that  $D_\sigma(0) = 0$ . Therefore  $z^*$  must be a maximum. This implies that there exist interior thresholds  $\Theta_{\sigma,1} \leq \Theta_{\sigma,2}$  such that  $D_\sigma(\cdot)$  is increasing on  $(-\infty, \Theta_{\sigma,1}]$  and decreasing on  $[\Theta_{\sigma,2}, 0]$ .<sup>30</sup> Finally, because  $a_1(q) \geq a_2(q)$  if and only if  $D'_\sigma(q) \geq 0$ , the desired result follows.  $\square$

*Proof of Proposition 2.* This proof is organized in 3 parts. I first show that the desired relationships hold with weak inequality. Then I show that they in fact hold with strict inequality.

**Part I:**  $\hat{a}(q) \geq a(q)$  for all  $q$ .

Note that  $c(a) = \frac{a^{p+1}}{p+1}$  implies that  $f(x) = x^{1/p}$  and  $c(f(x)) = \frac{x^{\frac{p+1}{p}}}{p+1}$ . As a result (4) and the first-best HJB equation can be written as

$$\begin{aligned} rJ(q) &= \left( n - \frac{1}{p+1} \right) [J'(q)]^{\frac{p+1}{p}} + \frac{\sigma^2}{2} J''(q) \text{ and} \\ r\hat{J}(q) &= \frac{p}{p+1} \left[ n\hat{J}'(q) \right]^{\frac{p+1}{p}} + \frac{\sigma^2}{2} \hat{J}''(q), \end{aligned}$$

respectively, where the subscript for the  $i^{\text{th}}$  agent has been suppressed since the equilibria are symmetric. Note that the equilibrium effort level of each agent is given by  $f(J'(q))$ , while the first-best effort level of each agent is given by  $f(n\hat{J}'(q))$ . Because  $f(\cdot)$  is strictly increasing, it suffices to show that  $n\hat{J}'(q) \geq J'(q)$  for all  $q$ . Let  $\alpha = \left[ \frac{np}{np+(n-1)} \right]^p n$ , and note that (i)  $\alpha \leq n$  for all  $p \geq 1$  and  $n \in \mathbb{N}$ , (ii)  $\alpha|_{n=1} = 1$ , and (iii)  $\alpha$  is strictly increasing in  $n$  for all  $n \geq 2$ . Therefore  $1 < \alpha \leq n$  for all  $p \geq 1$  and  $n \geq 2$ . Because  $\hat{J}'(q) > 0$  and  $J'(q) > 0$  for all  $q$ , it suffices to show that  $\alpha\hat{J}'(q) \geq J'(q)$  for all  $q$ . Now define  $\Delta_\alpha(q) = \alpha\hat{J}(q) - J(q)$  and note that  $\Delta_\alpha(\cdot)$  is smooth,  $\lim_{q \rightarrow -\infty} \Delta_\alpha(q) = 0$ , and  $\Delta_\alpha(0) = (\alpha - 1)V > 0$ . So either  $\Delta_\alpha(\cdot)$  is increasing on  $(-\infty, 0]$  or it has at least one interior global extreme point. If the former is true, then the desired inequality holds. Now suppose the latter is true and let us

<sup>30</sup>Unfortunately, it is not possible to prove that  $D_\sigma(\cdot)$  does not have any local extrema so that  $\Theta_{\sigma,1} = \Theta_{\sigma,2}$ , which would imply that  $a_1(q) \geq a_2(q)$  if and only if  $q \leq \Theta_{\sigma,i}$ .

denote this extreme point by  $z^*$ . Using that  $\alpha \hat{J}'(z^*) = J'(z^*)$ , (4) and the first-best HJB equation, we have that

$$\begin{aligned} r\Delta_\alpha(z^*) &= \left[ \frac{\alpha p}{p+1} \left(\frac{n}{\alpha}\right)^{\frac{p+1}{p}} - n + \frac{1}{p+1} \right] [J'(q)]^{\frac{p+1}{p}} + \frac{\sigma^2}{2} \Delta_\alpha''(z^*) \\ \implies r\Delta_\alpha(z^*) &= \frac{\sigma^2}{2} \Delta_\alpha''(z^*) . \end{aligned}$$

<sup>31</sup> Suppose that  $z^*$  is a global maximum. Then  $\Delta_\alpha''(z^*) \leq 0$  implies that  $\Delta_\alpha(z^*) \leq 0$ , contradicting the fact that  $\Delta_\alpha(0) > 0$ . Therefore,  $z^*$  must be a minimum. Then  $\Delta_\alpha''(z^*) \geq 0$  implies that  $\Delta_\alpha(z^*) \geq 0$ , contradicting the facts that  $\lim_{q \rightarrow -\infty} \Delta_\alpha(q) = 0$  and that  $z^*$  is interior. Therefore  $\Delta_\alpha(\cdot)$  cannot have any interior extreme points, which implies that  $\Delta_\alpha(\cdot)$  is increasing on  $(-\infty, 0]$ .

**Part II:**  $\hat{J}(q) \geq J(q)$  for all  $q$ .

Let us define  $\Delta_1(q) = \hat{J}(q) - J(q)$  and note that  $\Delta_1(\cdot)$  is smooth, and  $\lim_{q \rightarrow -\infty} \Delta_1(q) = \Delta_1(0) = 0$ . Therefore either  $\Delta_1(\cdot) \equiv 0$ , or  $\Delta_1(\cdot)$  has at least one local interior extreme point. If the former is true, then  $\Delta_1'(q) = \Delta_1''(q) = 0$  for all  $q$ . Then using (4) and the first-best HJB equation, it follows that  $\frac{1}{p+1} \left[ pn^{\frac{p+1}{p}} - n(p+1) + 1 \right] [J'(q)]^{\frac{p+1}{p}} = 0$ , which contradicts the facts that  $J'(z^*) > 0$  and  $\left[ pn^{\frac{p+1}{p}} - n(p+1) + 1 \right] > 0$  for all  $n \geq 2$  and  $p \geq 1$ . Therefore it must be the case that  $\Delta_1(\cdot)$  has an interior extreme point, which we denote by  $z^*$ . Using that  $\hat{J}'(z^*) = J'(z^*)$ , (4) and the first-best HJB equation, we have that

$$r\Delta_1(z^*) = \frac{pn^{\frac{p+1}{p}} - n(p+1) + 1}{p+1} [J'(z^*)]^{\frac{p+1}{p}} + \frac{\sigma^2}{2} \Delta_1''(z^*) .$$

Suppose that  $z^*$  is a minimum. Then  $\Delta_1''(z^*) \geq 0$  and  $\left[ pn^{\frac{p+1}{p}} - n(p+1) + 1 \right] > 0$  implies that  $\Delta_1(z^*) > 0$ , which in turn implies that  $\Delta_1(q) \geq 0$ , or equivalently  $\hat{J}(q) \geq J(q)$  for all  $q$ .

**Part III:**  $\hat{a}(q) > a(q)$  and  $\hat{J}(q) > J(q)$  for all  $q$ .

Recall that in proving existence of a MPE in Theorem 1 (Part I), I obtained a bound  $|J''(q)| \leq C[|J(q)| + |J'(q)|]$  for all  $q$ , where  $C > 0$  is a constants. Using an analogous approach, one can obtain a similar bound for  $|\hat{J}''(q)|$ ; i.e.,  $|\hat{J}''(q)| \leq \hat{C} \left[ |\hat{J}(q)| + |\hat{J}'(q)| \right]$  for all  $q$ .

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<sup>31</sup>Note that the constant  $\alpha$  has been chosen such that the term in brackets equals 0 when  $\alpha \hat{J}'(z^*) = J'(z^*)$ .

Suppose that there exists a  $z \leq 0$  such that  $\Delta'_\alpha(z) = 0$ . Because  $r\Delta_\alpha(z) = \frac{\sigma^2}{2}\Delta''_\alpha(z)$ , using the same argument used to establish Proposition 1 (ii), it follows that  $z$  must be a minimum such that  $\Delta_\alpha(z) = 0$ , and  $\Delta_\alpha(q) = 0$  for all  $q \leq z$ . The last equality implies that  $\Delta'_\alpha(q) = 0$  for all  $q < z$ . Now define  $M_\alpha(q) = \alpha \left[ \hat{J}(q) + \hat{J}'(q) \right] - [J(q) + J'(q)]$ , and note by parts I and II that  $M_\alpha(q) \geq 0$  for all  $q$ . Also  $M_\alpha(q) = 0$  for all  $q < z$ , and there exists a constant  $C_\alpha > 0$  such that  $M'_\alpha(q) \leq C_\alpha \cdot M_\alpha(q)$  for all  $q$ . By applying the differential form of Grönwall's inequality, it follows that  $M_\alpha(q) = 0$  for all  $q$ . However this contradicts the facts that  $\alpha\hat{J}(0) - J(0) > 0$  and  $\alpha\hat{J}'(0) \geq J'(0)$ . Therefore there does not exist a  $z$  such that  $\Delta'_\alpha(z) = 0$ , so that  $\alpha\hat{J}'(q) > J'(q)$  for all  $q$ , which implies that  $\hat{a}(q) > a(q)$  for all  $q$ .

To show that  $\hat{J}(q) > J(q)$  for all  $q$ , I use the same approach as above. First note that if there exists a  $\hat{z} < 0$  such that  $\Delta_1(\hat{z}) = 0$ , then  $\Delta_1(q) = 0$  for all  $q \leq \hat{z}$ . Then by defining  $M(q) = \left[ \hat{J}(q) + \hat{J}'(q) \right] - [J(q) + J'(q)]$ , and by using the fact that  $M(q) > 0$  for at least some  $q$ , and the differential form of Grönwall's inequality, the desired result follows. The details are omitted. □

*Proof of Theorem 2.* This proof is organized in 4 parts.

**Proof for (A) under Public Good Allocation:**

To begin let us define  $D_{n,m}(q) = J_m(q) - J_n(q)$ , and note that  $D_{n,m}(q)$  is smooth, and  $D_{n,m}(0) = \lim_{q \rightarrow -\infty} D_{n,m}(q) = 0$ . Therefore, either  $D_{n,m}(\cdot) \equiv 0$ , or it has an interior extreme point. Suppose the former is true. Then  $D_{n,m}(\cdot) \equiv D'_{n,m}(\cdot) \equiv D''_{n,m}(\cdot) \equiv 0$  together with (4) implies that  $f(J'_n(q))J'_n(q) = 0$  for all  $q$ . However, this contradicts Theorem 1 (ii), so that  $D_{n,m}(\cdot)$  must have an interior extreme point, which I denote by  $z^*$ . Then  $D'_{n,m}(z^*) = 0 \Rightarrow J'_m(z^*) = J'_n(z^*)$ , and  $D''_{n,m}(z^*) \geq 0$ . By using (4) we have

$$rD_{n,m}(z^*) = \frac{\sigma^2}{2}D''_{n,m}(z^*) + (m-n)f(J'_n(z^*))J'_n(z^*) > 0 = rD_{n,m}(0),$$

which implies that  $z^*$  is either a global maximum, or a local extreme point satisfying  $D_{n,m}(z^*) \geq 0$ . Therefore,  $J_m(q) \geq J_n(q)$  (i.e.,  $D_{n,m}(q) \geq 0$ ) for all  $q$ .

I now show that  $D_{n,m}(q)$  is single-peaked. Suppose it is not. Then there must exist a local maximum  $z^*$  followed by a local minimum  $\bar{z} > z^*$ . Clearly,  $D_{n,m}(\bar{z}) < D_{n,m}(z^*)$ ,  $D'_{n,m}(\bar{z}) = D'_{n,m}(z^*) = 0$ ,  $D''_{n,m}(\bar{z}) \geq 0 \geq D''_{n,m}(z^*)$ , and by Theorem 1 (iii),  $J'_n(\bar{z}) > J'_n(z^*)$ .

By using (4), at  $\bar{z}$  we have

$$\begin{aligned} rD_{n,m}(\bar{z}) &= \frac{\sigma^2}{2} D''_{n,m}(\bar{z}) + (m-n) f(J'_m(\bar{z})) J'_m(\bar{z}) \\ &> \frac{\sigma^2}{2} D''_{n,m}(z^*) + (m-n) f(J'_m(z^*)) J'_m(z^*) = rD_{n,m}(z^*), \end{aligned}$$

which contradicts the assumption that  $z^*$  is a local maximum and  $\bar{z}$  is a local minimum. By noting that  $D_{n,m}(\cdot)$  cannot be strictly increasing or strictly decreasing due to the boundary conditions, it follows that  $D_{n,m}(\cdot)$  is single-peaked; i.e., there exists a  $\Theta_{n,m} \leq 0$  such that  $J'_m(q) \geq J'_n(q)$  (because  $D'_{n,m}(q) \geq 0$ ), and consequently  $a_m(q) > a_n(q)$ , if and only if  $q \leq \Theta_{n,m}$ .

### Proof for (A) under Budget Allocation

Recall that under the public good allocation scheme, we had the boundary condition  $D_{n,m}(0) = 0$ . This condition is now replaced by  $D_{n,m}(0) = \frac{V}{m} - \frac{V}{n} < 0$ . Therefore,  $D_{n,m}(\cdot)$  is either decreasing, or it has at least one extreme point. Using similar arguments as above, it follows that any extreme point  $z^*$  is a global maximum and  $D_{n,m}(\cdot)$  may be at most single-peaked. Hence either  $D_{n,m}(\cdot)$  is decreasing in which case  $\Theta_{n,m} = -\infty$ , or there exists an interior  $\Theta_{n,m}$  such that  $a_m(q) \geq a_n(q)$  if and only if  $q \leq \Theta_{n,m}$ . The details are omitted.

### Proof for (B) under Public Good Allocation:

Note that  $c(a) = \frac{a^{p+1}}{p+1}$  implies that  $f(x) = x^{1/p}$  and  $c(f(x)) = \frac{x^{\frac{p+1}{p}}}{p+1}$ . As a result, (4) can be written for an  $n$  – member team as

$$rJ_n(q) = \left( n - \frac{1}{p+1} \right) (J'_n(q))^{\frac{p+1}{p}} + \frac{\sigma^2}{2} J''_n(q). \quad (9)$$

To compare the total effort of the team at every state of the project, we need to compare  $mf(J'_m(q))$  and  $nf(J'_n(q))$ , or equivalently  $(m^p J'_m(q))^{1/p}$  and  $(n^p J'_n(q))^{1/p}$ . Define  $\bar{D}_{n,m}(q) = m^p J'_m(q) - n^p J'_n(q)$ , and observe that  $\bar{D}'_{n,m}(q) \geq 0 \iff ma_m(q) \geq na_n(q)$ . Note that  $\bar{D}_{n,m}(0) = (m^p - n^p)V > 0$  and  $\lim_{q \rightarrow -\infty} \bar{D}_{n,m}(q) = 0$ . As a result, either  $\bar{D}_{n,m}(q)$  is increasing for all  $q$ , which implies that  $ma_m(q) \geq na_n(q)$  for all  $q$  and hence  $\Phi_{n,m} = 0$ , or  $\bar{D}_{n,m}(q)$  has an interior extreme point  $z^*$ . Suppose the latter is true. Then  $\bar{D}'_{n,m}(z^*) = 0$  implies that  $J'_m(z^*) = \left(\frac{n}{m}\right)^p J'_n(z^*)$ . Multiplying both sides of (9) by  $m^p$  and  $n^p$  for  $J_m(\cdot)$  and  $J_n(\cdot)$ , respectively, and subtracting the two quantities yields

$$r\bar{D}_{n,m}(z^*) = -\frac{n^p}{p+1} \left[ \frac{n}{m} - 1 \right] (J'_n(z^*))^{\frac{p+1}{p}} + \frac{\sigma^2}{2} \bar{D}''_{n,m}(z^*),$$

and observe that the first term in the RHS is strictly positive. Now suppose  $z^*$  is a global minimum. Then  $\bar{D}''_{n,m}(z^*) \geq 0$ , which implies that  $\bar{D}_{n,m}(z^*) > 0$ , but this contradicts the facts that  $\lim_{q \rightarrow -\infty} \bar{D}_{n,m}(q) = 0$  and  $z^*$  is interior. Hence  $z^*$  must be a global maximum or a local extreme point satisfying  $\bar{D}_{n,m}(z^*) \geq 0$ .

To complete the proof for this case, I now show that  $\bar{D}_{n,m}(\cdot)$  can be at most single-peaked. Suppose that the contrary is true. Then there exists a local maximum  $z^*$  followed by a local minimum  $\bar{z} > z^*$ . Because  $\bar{D}'_{n,m}(z^*) = \bar{D}'_{n,m}(\bar{z}) = 0$ ,  $\bar{D}''_{n,m}(\bar{z}) \geq 0 \geq \bar{D}''_{n,m}(z^*)$ , and by Theorem 1 (iii)  $J'_n(z^*) < J'_n(\bar{z})$ , it follows that  $\bar{D}_{n,m}(z^*) < \bar{D}_{n,m}(\bar{z})$ . However, this contradicts the facts that  $z^*$  is a local maximum and  $\bar{z}$  is a local minimum, which implies that  $\bar{D}_{n,m}(\cdot)$  is either strictly increasing in which case  $\Phi_{n,m} = 0$ , or it has a global interior maximum and no other local extreme points, in which case there exists an interior  $\Phi_{n,m}$  such that  $m a_m(q) \geq n a_n(q)$  if and only if  $q \leq \Phi_{n,m}$ .

### Proof for (B) under Budget Allocation

The only difference compared to the proof under public good allocation is the boundary condition at 0; i.e.,  $\bar{D}_{n,m}(0) = m^p J_m(0) - n^p J_n(0) = (m^{p-1} - n^{p-1})V > 0$  (recall  $p \geq 1$ ). As a result, the same proof applies. Note that if  $p = 1$  (i.e., effort costs are quadratic), then  $\bar{D}_{n,m}(0) = 0$  and hence  $\Phi_{n,m}$  is interior. □

*Proof of Proposition 3.* Let us first consider the statement under public good allocation. In the proof for statement (A) of Theorem 2, I showed that  $D_{n,n+1}(q) = J_{n+1}(q) - J_n(q) \geq 0$  for all  $q$ . This implies that  $J_{n+1}(q_0) \geq J_n(q_0)$  for all  $q_0 \leq 0$ .

Now consider the statement under budget allocation. In the proof for statement (A) of Theorem 2, I showed that  $D_{n,n+1}(\cdot) = J_{n+1}(\cdot) - J_n(\cdot)$  is either decreasing, or it has exactly one extreme point which must be a maximum. Moreover,  $\lim_{q \rightarrow -\infty} D_{n,n+1}(q) = 0$  and  $D_{n,n+1}(0) < 0$ . This implies that there exists a threshold  $T_n$  (may be  $-\infty$ ) such that  $J_{n+1}(q_0) \geq J_n(q_0)$  if and only if  $q_0 \leq -T_n$ , or equivalently if and only if  $|q_0| \geq T_n$ . □

*Proof of Theorem 3.* This proof is organized in 5 parts. I first show that a solution to (5) subject to the boundary conditions (6) exists. Then I show that properties (i) and (ii) hold. Finally, I show that the solution to the above boundary value problem is unique. The proofs resemble those in Theorem 1 closely.

**Part I:** Existence of a solution.

First note that  $J_i(\cdot)$  depends only on  $V_i$  for all  $i$  and not on  $F(\cdot)$ , so for given  $V_i$  I can solve  $F(\cdot)$  by taking  $J_i(\cdot)$  as given for all  $i$ . I shall use a similar approach as that used to prove existence for  $J_i(\cdot)$ . Note that (5) and (6) can be re-written as

$$F_N''(q) = \frac{2r}{\sigma^2} F_N(q) + \frac{2}{\sigma^2} \left[ \sum_{i=1}^n f(J_i'(q)) \right] F_N'(q) \quad (10)$$

subject to  $F_N(-N) = 0$  and  $F_N(0) = F_0$ ,

where  $F_0 = U - \sum_{i=1}^n V_i > 0$ . Let  $h(F_N, F_N')$  denote the RHS of (10), and observe that  $h(\cdot, \cdot)$  is continuous. Now fix some arbitrary  $K > 0$  and define a new function

$$h_K(F_N, F_N') = \max \{ \min \{ h(F_N, F_N'), K \}, -K \}.$$

Note that  $h_K(\cdot, \cdot)$  is continuous and bounded, so that by the Scorza-Dragoni Lemma (see Lemma 4 in Hartman (1960)), there exists a solution to  $F_{N,K}'' = h_K(F_{N,K}, F_{N,K}') [-N, 0]$  subject to  $F_{N,K}(-N) = 0$  and  $F_{N,K}(0) = F_0$ . The next part of the proof involves showing that there exists some  $\bar{K}$  such that  $h_K(F_{N,K}, F_{N,K}') \in [-\bar{K}, \bar{K}]$  for all  $K$  on  $[-N, 0]$ , which will imply that the solution  $F_{N,\bar{K}}''(\cdot)$  satisfies (10). The final step involves showing that a solution exists when  $N \rightarrow \infty$ , so that a solution to (5) subject to (6) exists.

By part I of Theorem 1, there exists an  $\bar{A}$  such that  $|J_i'(q)| \leq \bar{A}$  for all  $q$ , and it is straightforward to show that  $F_{N,K}(q) \in [0, F_0]$  and  $F_{N,K}'(q) \geq 0$  for all  $q$ . As a result, letting  $\Omega = nf(\bar{A})$ , a bound for  $|F_{N,K}''(q)|$  can be obtained by

$$|F_{N,K}''(q)| \leq \frac{2r}{\sigma^2} F_0 + \frac{2}{\sigma^2} \Omega F_{N,K}'(q).$$

By noting that  $F_N(0) > 0$  and using the mean-value theorem, it follows that there exists a  $z^* \in [-N, 0]$  such that  $F_N'(z^*) = \frac{F_0}{N}$ . Hence, for all  $z \in [-N, 0]$

$$F_0 > \left| \int_{z^*}^z F_N'(q) dq \right| \geq \frac{\sigma^2}{2} \left| \int_{z^*}^z F_N'(q) \frac{F_N''(q)}{rF_0 + \Omega F_N'(q)} dq \right| \geq \frac{\sigma^2}{2} \left| \int_0^{F_N'(z)} \frac{s}{rF_0 + \Omega s} ds \right|,$$

where I let  $s = F_N'(q)$  and used that  $F_N'(q) F_N''(q) = F_N'(q) dF_N'(q)$ . The fact that  $\int_0^\infty \frac{s}{rF_0 + \Omega s} ds = \infty$  implies that there exists a  $\bar{B} < \infty$  such that  $\frac{\sigma^2}{2} \left| \int_0^{\bar{B}} \frac{s}{rF_0 + \Omega s} ds \right| = F_0$ . This implies that  $F_N'(q) \leq \bar{B}$  for all  $q \in [-N, 0]$ .

Because  $\bar{B}$  is independent of both  $N$  and  $K$ ,  $F'_{N,K}(q) \in [0, \bar{B}]$  for all  $q \in [-N, 0]$ ,  $N \in \mathbb{N}$ , and  $K > 0$ . In addition, we now that  $F_{N,K}(q) \in [0, F_0]$  for all  $q \in [-N, 0]$ ,  $N \in \mathbb{N}$ , and  $K > 0$ . Now let  $\bar{K} = \frac{2r}{\sigma^2}F_0 + \frac{2}{\sigma^2}\Omega\bar{B}$ , and observe that a solution to  $F''_{N,\bar{K}} = h_{\bar{K}}(F_{N,\bar{K}}, F'_{N,\bar{K}})$  subject to  $F_{N,\bar{K}}(-N) = 0$  and  $F_{N,\bar{K}}(0) = F_0$  exists, and  $h_{\bar{K}}(F_{N,\bar{K}}(q), F'_{N,\bar{K}}(q)) = h(F_{N,\bar{K}}(q), F'_{N,\bar{K}}(q))$  for all  $q \in [-N, 0]$ . Therefore,  $F_{N,\bar{K}}(\cdot)$  solves (10).

To show that a solution for (10) as  $N \rightarrow \infty$  exists, recall that there exists a constant  $\bar{B}$  such that  $|F'_N(q)| \leq \bar{B}$  on  $[-N, 0]$  for all  $N \in \mathbb{N}$ . Hence the sequences  $\{F_N(\cdot)\}$  and  $\{F'_N(\cdot)\}$  are uniformly bounded and equicontinuous on  $[-N, 0]$ . By applying the Arzela-Ascoli theorem to a sequence of intervals  $[-N, 0]$  and letting  $N \rightarrow \infty$ , it follows that the system of ODE defined by (5) subject to (6) has at least one solution.

**Part II:**  $F(q) > 0$  for all  $q$ .

First note that  $\lim_{q \rightarrow -\infty} F(q) = 0$  and  $F(0) > 0$ . Let  $z^* = \arg \min_{q \leq 0} \{F(q)\}$ . Clearly,  $z^* < 0$ . If  $z^* = -\infty$ , then together with the fact that  $\lim_{q \rightarrow -\infty} F(q) = 0$ , this implies that  $F(q) > 0$  for all  $q$ , which proves the desired statement. So suppose that  $z^*$  is interior. Then  $F'(z^*) = 0$  and  $F''(z^*) \geq 0$ , which using (5) implies that  $F(z^*) \geq 0$ . Because  $\lim_{q \rightarrow -\infty} F(q) = 0$ , it follows that  $F(z^*) = 0$ . Now suppose that  $F(q) \neq 0$  for at least some  $q$ . Then there exists some  $\bar{z}$  such that  $F'(\bar{z}) = 0$ , which using (5) implies that  $rF(\bar{z}) = \frac{\sigma^2}{2}F''(\bar{z})$ . By noting that any maximum must satisfy  $F''(\bar{z}) \leq 0 \implies F(\bar{z}) \leq 0$ , while any minimum must satisfy  $F''(\bar{z}) \geq 0 \implies F(\bar{z}) \geq 0$ , it follows that if there exists a  $z^*$  such that  $F(z^*) = 0$ , then  $F(q) = 0$  and  $F'(q) = 0$  for all  $q < z^*$ . By applying the differential form of Grönwall's inequality to  $|F(q)| + |F'(q)|$  and using that  $|F''(q)| \leq \frac{2r}{\sigma^2}|F(q)| + \frac{2nf(\bar{A})}{\sigma^2}|F'(q)|$ , it follows that  $F(q) = 0$  for all  $q$ . However this contradicts the fact that  $F(0) > 0$ . Hence  $F(\cdot)$  cannot have an interior minimum, and there cannot exist an interior  $z^*$  such that  $F(z^*) = 0$ . Hence  $F(q) > 0$  for all  $q$ .

**Part III:**  $F'(q) > 0$  for all  $q$ .

Because  $F(\cdot)$  is continuous and  $\lim_{q \rightarrow -\infty} F(q) = 0 < F(0)$ , there exists a  $-\infty < \Lambda < 0$  such that  $F(\Lambda) < F(0)$ , and by the mean-value theorem, there exists a  $z^* \in (\Lambda, 0)$  such that  $F'(z^*) = \frac{F(0) - F(\Lambda)}{-\Lambda} > 0$ . Suppose that there exists a  $z^{**}$  such that  $F'(z^{**}) \leq 0$ . Then by the intermediate value theorem, there exists a  $\bar{z}$  between  $z^*$  and  $z^{**}$  such that  $F'(\bar{z}) = 0$ . Using (5) and the fact that  $F(q) > 0$  for all  $q$ , it follows that  $rF(\bar{z}) = \frac{\sigma^2}{2}F''(\bar{z}) > 0$ ; i.e.,  $\bar{z}$  is a minimum. Because  $\bar{z}$  is interior,  $\lim_{q \rightarrow -\infty} F(q) = 0$ , and  $F(\bar{z}) > 0$ , there exists an interior local maximum  $\hat{z} < \bar{z}$ , so that  $F'(\hat{z}) = 0$  and  $F''(\hat{z}) \leq 0$ . Using (5), it follows that

$F(\hat{z}) \leq 0$ , which contradicts the fact that  $F(q) > 0$  for all  $q$ . Therefore,  $F'(q) > 0$  for all  $q$ .

**Part IV:**  $F(q)$  is infinitely differentiable on  $(-\infty, 0]$ .

By noting that  $\lim_{q \rightarrow -\infty} F(q) = \lim_{q \rightarrow -\infty} F'(q) = 0$ , and by twice integrating both sides of (5) over the interval  $(-\infty, q]$ , we have that

$$F(q) = \int_{-\infty}^q \int_{-\infty}^y \frac{2r}{\sigma^2} F(z) + \frac{2}{\sigma^2} \left[ \sum_{i=1}^n f(J'_i(z)) \right] F'(z) dz dy.$$

Recall that  $f(x) = x^{1/p}$  and  $J'_i(q) > 0$  for all  $q$ . Since a solution  $F(\cdot)$  satisfying (5) subject to the boundary conditions (6) exists,  $F(q)$  and  $F'(q)$  are continuous. As a result, the function under the integral is continuous and infinitely differentiable in  $F(z)$ ,  $F'(z)$  and  $J'_i(z)$  for all  $i$ . By noting that  $F(q)$  is differentiable twice more than the function under the integral, using Theorem 1 (iv), and proceeding by induction, property (iii) is proven.

**Part V:** Uniqueness of a solution.

Because  $F(\cdot)$  is a function of  $J'_i(\cdot)$  for all  $i$ , and Theorem 1 established that  $J_i(\cdot)$  is unique if the team comprises of  $n$  symmetric, or 2 asymmetric agents, I focus only on these cases only. Suppose that there exist two solutions that solve (5) subject to the initial conditions (6), denoted by  $F_1(\cdot)$  and  $F_2(\cdot)$ , respectively. Let  $\Delta F(q) = F_1(q) - F_2(q)$ , and note that  $\Delta F(0) = \lim_{q \rightarrow -\infty} \Delta F(q) = 0$ , and  $\Delta F(\cdot)$  is smooth. Also observe that either  $\Delta F(\cdot) \equiv 0$ , or  $\Delta F(\cdot)$  has a global extreme point. Suppose the latter is true and letting  $z^*$  be such extreme point, we have that  $\Delta F'(z^*) = 0$ . Using (5) and the facts that  $\Delta F''(z^*) \geq 0$  if  $z^*$  is a minimum and  $\Delta F''(z^*) \leq 0$  if  $z^*$  is a maximum, it follows that  $\Delta F(q) = 0$  for all  $q$ . Hence  $F_1(\cdot) \equiv F_2(\cdot)$  and the proof is complete. □

*Proof of Theorem 4.* To prove this result, first fix a set of arbitrary milestones  $Q_1 < \dots < Q_K = 0$  where  $K$  is arbitrary but finite, and assume that the manager allocates budget  $w_k > 0$  for compensating the agents upon reaching milestone  $k$  for the first time. Now consider the following compensation schemes. Let  $B = \sum_{k=1}^K w_k$ . Under *scheme (a)*, each agent is paid  $\frac{B}{n}$  upon completion of the project and receives no intermediate compensation while the project is in progress. Under *scheme (b)*, each agent is paid  $\frac{w_k}{n \mathbb{E}_{\tau_k} [e^{r\tau_k} | Q_i]}$  when  $q_t$  hits  $Q_k$  for the first time, where  $\tau_k$  denotes the random time to completion given that the current state of the project is  $Q_k$ . I shall show that the manager is always better off using scheme (a) relative to scheme (b).

Some remarks are in order. First, note that scheme (b) ensures that the expected total cost for compensating each agent equals  $\frac{B}{n}$  to facilitate comparison between the two schemes. Second, observe that while the expected total cost for compensating the agents is the same under the two schemes, the associated variance is zero under scheme (a), while it is strictly positive under scheme (b) due to the stochastic evolution of the project. Therefore, if the manager is credit constrained or ambiguity / risk averse, then scheme (a) is favored even more. Third, since the manager values the project at  $U$ , without loss of generality, I can restrict attention to allocations  $\{w_k\}_{k=1}^K$  such that  $\sum_{k=1}^K w_k = B < U$ .

This proof is organized in 3 parts. In **part I**, I introduce the necessary functions (i.e., ODEs) that will be necessary for the proof. In **part II**, I show that each agent exerts higher effort under scheme (a) relative to scheme (b). Finally, in **part III**, I show that the manager's expected discounted profit is higher under scheme (a) relative to scheme (b) for any choice of  $Q_k$ 's and  $w_k$ 's.

**Part I:** To begin, I introduce the expected discounted payoff and discount rate functions (i.e., ODEs) that will be necessary for the proof. Under *scheme (a)*, given the current state  $q$ , each agent's expected discounted payoff satisfies

$$rJ(q) = -c(f(J'(q))) + nf(J'(q))J'(q) + \frac{\sigma^2}{2}J''(q) \text{ subject to } \lim_{q \rightarrow -\infty} J(q) = 0 \text{ and } J(0) = \frac{B}{n}.$$

On the other hand, under *scheme (b)*, given the current state  $q$  and that  $k-1$  milestones have been reached, each agent's expected discounted payoff, which is denoted by  $J_k(q)$ , satisfies

$$rJ_k(q) = -c(f(J'_k(q))) + nf(J'_k(q))J'_k(q) + \frac{\sigma^2}{2}J''_k(q) \text{ on } (-\infty, Q_k]$$

subject to

$$\lim_{q \rightarrow -\infty} J_k(q) = 0 \text{ and } J_k(Q_k) = \frac{w_k}{n\mathbb{E}_{\tau_k}[e^{r\tau_k} | Q_k]} + J_{k+1}(Q_k),$$

where  $J_{K+1}(Q_K) = 0$ .<sup>32</sup> The second boundary condition states that upon reaching milestone  $Q_k$  for the first time, each agent is paid  $\frac{w_k}{n\mathbb{E}_{\tau_k}[e^{r\tau_k} | Q_k]}$ , and he receives the continuation value  $J_{k+1}(Q_k)$  from future progress. Eventually upon reaching the  $K^{th}$  milestone, the project is completed so that each agent is paid  $\frac{w_K}{n}$ , and receives no continuation value. Note that due to the stochastic evolution of the project, even after the  $k^{th}$  milestone has been reached for the first time, the state of the project may drift below  $Q_k$ . Therefore, the first boundary

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<sup>32</sup>Since this proof considers a fixed team size  $n$ , we use to subscript  $k$  to denote that  $k-1$  milestones have been reached.

condition ensures that as  $q \rightarrow -\infty$ , the expected time until the project is completed so that each agent collects his reward diverges to  $\infty$ , which together with the fact that  $r > 0$ , implies that his expected discounted payoff asymptotes to 0. Using the same approach as used in Theorem 1, it is straightforward to show that for each  $k$ ,  $J_k(\cdot)$  exists, it is unique, smooth, strictly positive, strictly increasing and strictly convex on its domain.

Next, let us denote the expected *discount rate* until the project is completed under scheme (a), given the current state  $q$ , by  $T(q) = \mathbb{E}_\tau [e^{-r\tau} | q]$ . Using the same approach as used to derive the manager's HJB equation, it follows that

$$rT(q) = nf(J'(q))T'(q) + \frac{\sigma^2}{2}T''(q) \text{ subject to } \lim_{q \rightarrow -\infty} T(q) = 0 \text{ and } T(0) = 1.$$

The first boundary condition states that as  $q \rightarrow -\infty$ , the expected time until the project is completed diverges to  $\infty$ , so that  $\lim_{q \rightarrow -\infty} T(q) = 0$ . On the other hand, when the project is completed so that  $q = 0$ , then  $\tau = 0$  with probability 1, which implies that  $T(0) = 1$ .

Next, let us consider scheme (b). Similarly, we denote the expected *discount rate* until the project is completed, given the current state  $q$  and that  $k - 1$  milestones have been reached, by  $T_k(q) = \mathbb{E}_{\tau_k} [e^{-r\tau_k} | q]$ . Then, it follows that

$$rT_k(q) = nf(J'_k(q))T'_k(q) + \frac{\sigma^2}{2}T''_k(q) \text{ on } (-\infty, Q_k]$$

subject to

$$\lim_{q \rightarrow -\infty} T_k(q) = 0, T_k(Q_k) = T_{k+1}(Q_k) \text{ for all } k \leq n,$$

where  $T_{K+1}(Q_K) = 1$ . The first boundary condition has the same interpretation as above. The second boundary condition ensures value matching; i.e., that upon reaching milestone  $k$  for the first time,  $T_k(Q_k) = T_{k+1}(Q_k)$ . Using the same approach as used in Theorem 3, it is straightforward to show that  $T(\cdot)$  and for each  $k$ ,  $T_k(\cdot)$  exists, it is unique, smooth, strictly positive, and strictly increasing on its domain.

Note that by Jensen's inequality,  $\frac{1}{\mathbb{E}_{\tau_k}[e^{r\tau_k}]} \leq \mathbb{E}_{\tau_k}[e^{-r\tau_k}]$ .<sup>33</sup> Therefore, using this inequality, and the second boundary condition for  $J_k(\cdot)$ , it follows that  $J_k(Q_k) \leq \frac{w_k}{n}T_k(Q_k) + J_{k+1}(Q_k)$ .

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<sup>33</sup>Because  $e^{\pm rt}$  is convex, it follows that  $e^{r\mathbb{E}\tau_k} \leq \mathbb{E}[e^{r\tau_k}]$  and  $e^{-r\mathbb{E}\tau_k} \leq \mathbb{E}[e^{-r\tau_k}]$ . The second inequality can be re-written as  $\frac{1}{\mathbb{E}[e^{-r\tau_k}]} \leq e^{r\mathbb{E}\tau_k}$ , so that  $\frac{1}{\mathbb{E}_{\tau_k}[e^{r\tau_k}]} \leq \mathbb{E}_{\tau_k}[e^{-r\tau_k}]$ .

**Part II:** The next step of the proof is to show that for any  $k$ ,  $J(Q_k) \geq J_k(Q_k)$ , and as a consequence of Proposition 1 (i),  $J'(q) \geq J'_k(q)$  for all  $q \leq Q_k$ . This will imply that agents exert higher effort under scheme (a) at every state of the project. To proceed, let us define  $\Delta_k(q) = J(q) - J_k(q) - \frac{1}{n} \left( \sum_{i=1}^{k-1} w_i \right) T_k(q)$  on  $(-\infty, Q_k]$  for all  $k$ , and note that  $\lim_{q \rightarrow -\infty} \Delta_k(q) = 0$  and  $\Delta_k(\cdot)$  is smooth.

First, I consider on the case in which  $k = K$ , and then I proceed by backward induction. Noting that  $\Delta_K(Q_K) = 0$  (where  $Q_K = 0$ ), either  $\Delta_K(\cdot) \equiv 0$  on  $(-\infty, Q_K]$ , or  $\Delta_K(\cdot)$  has some interior global extreme point  $z$ . If the former is true, then  $\Delta_K(q) = 0$  for all  $q \leq Q_K$ , so that  $J(Q_K) \geq J_K(Q_K)$ . Now suppose that the latter is true. Then  $\Delta'_K(z) = 0$  so that

$$\begin{aligned} r\Delta_K(z) &= -c(f(J'(z))) + nf(J'(z))J'(z) + c(f(J'_K(z))) - nf(J'_K(z))J'_K(z) \\ &\quad - \left( \sum_{i=1}^{m-1} w_i \right) f(J'_K(z))T'_K(z) + \frac{\sigma^2}{2}\Delta''_K(z). \end{aligned}$$

Because  $\Delta'_K(z) = 0$  implies that  $\left( \sum_{i=1}^{k-1} w_i \right) T'_K(z) = n[J'(z) - J'_K(z)]$ , the above equation can be re-written as

$$\begin{aligned} r\Delta_K(z) &= c(f(J'_K(z))) - c(f(J'(z))) + nf(J'(z))J'(z) - nf(J'_K(z))J'(z) + \frac{\sigma^2}{2}\Delta''_K(z) \\ &= \left\{ \frac{[J'_K(z)]^{\frac{p+1}{p}} - [J'(z)]^{\frac{p+1}{p}}}{p+1} + n[J'(z)]^{\frac{p+1}{p}} - n[J'_K(z)]^{\frac{1}{p}}J'(z) \right\} + \frac{\sigma^2}{2}\Delta''_K(z). \end{aligned}$$

To show that the term in brackets is strictly positive, note that  $J(Q_K) > J_K(Q_K)$  so that  $J'(z) > J'_K(z)$  by Proposition 1 (i), and  $J'_K(z) > 0$ . Therefore, let  $x = \frac{J'_K(z)}{J'(z)}$ , where  $x < 1$ , and observe that the term in brackets is non-negative if and only if

$$\begin{aligned} n(p+1)[J'(z)]^{\frac{p+1}{p}} - [J'(z)]^{\frac{p+1}{p}} &\geq n(p+1)[J'_K(z)]^{\frac{1}{p}}J'(z) - [J'_K(z)]^{\frac{p+1}{p}} \\ \implies n(p+1) - 1 &\geq n(p+1)x^{\frac{1}{p}} - x^{\frac{p+1}{p}}. \end{aligned}$$

Because the RHS is strictly increasing in  $x$ , and it converges to the LHS as  $x \rightarrow 1$ , I conclude that the above inequality holds.

Suppose that  $z$  is a global minimum. Then  $\Delta''_K(z) \geq 0$  together with the fact that the term in brackets is strictly positive implies that  $\Delta_K(z) > 0$ . Therefore, any interior global minimum must satisfy  $\Delta_K(z) \geq 0$ , which in turn implies that  $\Delta_K(q) \geq 0$  for all  $q$ . As a result,  $\Delta_K(Q_{K-1}) \geq 0$  or equivalently  $J(Q_{K-1}) \geq J_K(Q_{K-1}) + \frac{1}{n} \left( \sum_{i=1}^{K-1} w_i \right) T_K(Q_{K-1})$ .

Now consider  $\Delta_{K-1}(\cdot)$ , and note that  $\lim_{q \rightarrow -\infty} \Delta_{K-1}(q) = 0$ . By using the last inequality, that  $J_{K-1}(Q_{K-1}) \leq \frac{w_{K-1}}{n} T_{K-1}(Q_{K-1}) + J_K(Q_{K-1})$ , and  $T_{K-1}(Q_{K-1}) = T_K(Q_{K-1})$ , it follows that

$$\Delta_{K-1}(Q_{K-1}) = J(Q_{K-1}) - J_{K-1}(Q_{K-1}) - \frac{1}{n} \left( \sum_{i=1}^{K-2} w_i \right) T_{K-1}(Q_{K-1}) \geq 0.$$

Therefore, either  $\Delta_{K-1}(\cdot)$  is increasing on  $(-\infty, Q_{K-1}]$ , or it has some interior global extreme point  $z < Q_{K-1}$  such that  $\Delta'_{K-1}(z) = 0$ . If the former is true, then  $\Delta_{K-1}(Q_{K-2}) \geq 0$ . If the latter is true, then by applying the same technique as above we can again conclude that  $\Delta_{K-1}(Q_{K-2}) \geq 0$ .

Proceeding inductively, it follows that for all  $k \in \{2, \dots, K\}$ ,  $\Delta_k(Q_{k-1}) \geq 0$  or equivalently  $J(Q_{k-1}) \geq J_k(Q_{k-1}) + \frac{1}{n} \left( \sum_{i=1}^{k-1} w_i \right) T_k(Q_{k-1})$  and using that  $J_{k-1}(Q_{k-1}) \leq \frac{w_{k-1}}{n} T_k(Q_{k-1}) + J_k(Q_{k-1})$ , it follows that  $J(Q_{k-1}) \geq J_{k-1}(Q_{k-1})$ . Finally, by using Proposition 1 (i), it follows that for all  $k$ ,  $J'(q) \geq J'_k(q)$  for all  $q \leq Q_k$ .

**Part III:** Given a fixed expected budget  $B$ , the manager's objective is to maximize  $\mathbb{E}_\tau [e^{-r\tau} | q_0]$  or equivalently  $T(q_0)$ , where  $\tau$  denotes the completion time of the project, and it depends on the agents' strategies, which themselves depend on the set of milestones  $\{Q_k\}_{k=1}^K$  and payments  $\{w_k\}_{k=1}^K$ . Since  $q_0 < Q_1 < \dots < Q_K$ , it suffices to show that  $T(q_0) \geq T_1(q_0)$  in order to conclude that given any arbitrary choice of  $\{Q_k, w_k\}_{k=1}^K$ , the manager is better off compensating the agents only upon completing the project relative to also rewarding them for reaching intermediate milestones.

Define  $D_k(q) = T(q) - T_k(q)$  on  $(-\infty, Q_k]$  for all  $k \in \{1, \dots, K\}$ , and note that  $D_k(\cdot)$  is smooth and  $\lim_{q \rightarrow -\infty} D_k(q) = 0$ . Let us begin with the case in which  $k = K$ . Note that  $D_K(Q_K) = 0$  (where  $Q_K = 0$ ). So either  $D_K(\cdot) \equiv 0$  on  $(-\infty, Q_K]$ , or  $D_K(\cdot)$  has an interior global extreme point  $\bar{z} < Q_K$ . Suppose that  $\bar{z}$  is a global minimum. Then  $D'_K(\bar{z}) = 0$  so that

$$rD_K(\bar{z}) = n[J'(\bar{z}) - J'_K(\bar{z})]T'(\bar{z}) + \frac{\sigma^2}{2}D''_K(\bar{z}).$$

Recall that  $J'(q) \geq J'_k(q)$  for all  $q \leq Q_k$  from part II. Since  $\bar{z}$  is assumed to be a minimum, it must be true that  $D''_K(\bar{z}) \geq 0$ , which implies that  $D_K(\bar{z}) \geq 0$ . Therefore, any interior global minimum must satisfy  $D_K(\bar{z}) \geq 0$ , which implies that  $D_K(q) \geq 0$  for all  $q \leq Q_K$ . As a result,  $T(Q_{K-1}) \geq T_K(Q_{K-1}) = T_{K-1}(Q_{K-1})$ .

Next, consider  $D_{K-1}(\cdot)$ , recall that  $\lim_{q \rightarrow -\infty} D_{K-1}(q) = 0$ , and note that the above inequality implies that  $D_{K-1}(Q_{K-1}) \geq 0$ . By using the same technique as above, it follows that  $T(Q_{K-2}) \geq T_{K-1}(Q_{K-2}) = T_{K-2}(Q_{K-2})$ , and proceeding inductively we obtain that  $D_1(q) \geq 0$  for all  $q \leq Q_1$  so that  $T(q_0) \geq T_1(q_0)$ . □

*Proof of Proposition 4.* In preparation, I establish a Lemma that ensures that the single-crossing property of Milgrom and Shannon (1994) is satisfied.

**Lemma 1.** *Suppose the manager employs  $n$  identical agents, each of whom receives  $\frac{B}{n}$  upon completion. Then for all  $\delta \in (0, U - B)$ , there exists a threshold  $T_\delta$  such that she is better off increasing each agent's reward by  $\frac{\delta}{n}$  so that each agent receives  $\frac{B+\delta}{n}$  if and only if the size of the project  $|q_0| \geq T_\delta$ .*

*Proof of Lemma 1.* Consider 2 teams each comprising of  $n$  symmetric agents. Upon completion of the project, each member of the first team receives a reward  $\frac{B}{n}$ , while each member of the second team receives a reward  $\frac{B+\delta}{n}$ , where  $\delta > 0$ . Let us denote each agent's expected discounted payoff and equilibrium effort level of the two teams given  $q$  by  $\{J_0(q), a_0(q)\}$  and  $\{J_\delta(q), a_\delta(q)\}$ , respectively. From Proposition 1 (i) we know that  $a_\delta(q) > a_0(q)$  for all  $q$ ; i.e., each agent's effort level is strictly increasing in his compensation. Abusing notation, let us denote the manager's expected discounted profit given  $q$  for the two cases by  $F_B(q)$  and  $F_{B+\delta}(q)$ , respectively. Now let  $\Delta_V(\cdot) = F_B(\cdot) - F_{B+\delta}(\cdot)$ , and observe that  $\lim_{q \rightarrow -\infty} \Delta_V(q) = 0 < \delta = \Delta_V(0)$ . Because  $\Delta_V(\cdot)$  is smooth, it is either increasing on  $(-\infty, 0]$ , or it has an interior global extreme point. Suppose the latter is true and denote that extreme point by  $\bar{z}$ . By using (5), it follows that

$$r\Delta_V(\bar{z}) = n[a_B(\bar{z}) - a_{B+\delta}(\bar{z})]F'_B(\bar{z}) + \frac{\sigma^2}{2}\Delta''_V(\bar{z}).$$

Because  $F'_B(\bar{z}) > 0$ ,  $a_B(\bar{z}) < a_{B+\delta}(\bar{z})$ ,  $\Delta_V(0) > 0$ , and  $\bar{z}$  is interior, it follows that  $\bar{z}$  must be a global minimum. By noting that any local maximum  $\hat{z}$  must satisfy  $\Delta_V(\hat{z}) \leq 0$ , it follows that  $\Delta_V(\cdot)$  is either increasing on  $(-\infty, 0]$ , or it crosses 0 exactly once from below. Therefore there exists a  $T_\delta$  such that  $\Delta_V(q_0) \leq 0$  if and only if  $q_0 \leq -T_\delta$ , or equivalently, the manager is better off increasing each agent's reward by  $\frac{\delta}{n}$  if and only if  $|q_0| \geq T_\delta$ . By noting that  $T_\delta = -\infty$  if  $\Delta_V(\cdot)$  is increasing on  $(-\infty, 0]$ , the proof is complete. □

Other things equal, the manager chooses her budget  $B \in [0, U]$  to maximize her expected discounted profit at  $q_0$ ; i.e., she chooses  $B(|q_0|) = \arg \max_{B \in [0, U]} \{F_n(q_0; B)\}$ . By noting that the necessary conditions for the Monotonicity Theorem (i.e., Theorem 4) of Milgrom and Shannon (1994) to hold are satisfied, it follows that the manager's optimal budget  $B(|q_0|)$  is (weakly) increasing in the project size  $|q_0|$ . □

*Proof of Proposition 5.* Let us denote the manager's expected discounted profit when she employs  $n$  (symmetric) agents by  $F_n(\cdot)$ , and note that  $\lim_{q \rightarrow -\infty} F_n(q) = 0$  and  $F_n(0) = U - V > 0$  for all  $n$ . Now let us define  $\Delta_{n,m}(\cdot) = F_m(\cdot) - F_n(\cdot)$  and note that  $\Delta_{n,m}(\cdot)$  is smooth and  $\lim_{q \rightarrow -\infty} \Delta_{n,m}(q) = \Delta_{n,m}(0) = 0$ . It suffices to show that for all  $n$  and  $m$  there exists a  $T_{n,m} \leq 0$  such that  $F_m(q_0) \geq F_n(q_0)$  if and only if  $q_0 \leq T_{n,m}$ . Note that either  $\Delta_{n,m}(\cdot) \equiv 0$ , or  $\Delta_{n,m}(\cdot)$  has at least one global extreme point. Suppose that the former is true. Then  $\Delta_{n,m}(q) = \Delta'_{n,m}(q) = \Delta''_{n,m}(q) = 0$  for all  $q$ , together with (5), implies that  $[A_m(q) - A_n(q)] F'_n(q) = 0$  for all  $q$ , where  $A_n(\cdot) \equiv na_n(\cdot)$ . However, this is a contradiction, because  $A_m(q) > A_n(q)$  for at least some  $q$  by Theorem 2 (B), and  $F'_n(q) > 0$  for all  $q$  by Theorem 3 (i). Therefore,  $\Delta_{n,m}(\cdot)$  has at least one global extreme point, which I denote by  $\bar{z}$ . By using that  $\Delta'_{n,m}(\bar{z}) = 0$  and (5), we have that

$$r\Delta_{n,m}(\bar{z}) = [A_m(\bar{z}) - A_n(\bar{z})] F'_n(\bar{z}) + \frac{\sigma^2}{2} \Delta''_{n,m}(\bar{z}).$$

Recall that  $F'_n(\bar{z}) > 0$ , and from Theorem 2 (B) that for each  $n$  and  $m$  there exists an (interior) threshold  $\Phi_{n,m}$  such that  $A_m(q) \geq A_n(q)$  if and only if  $q \leq \Phi_{n,m}$ . It follows that  $\bar{z}$  is a global maximum if  $\bar{z} \leq \Phi_{n,m}$ , while it is a global minimum if  $\bar{z} \geq \Phi_{n,m}$ . Next observe that if  $\bar{z} \leq \Phi_{n,m}$  then any local minimum must satisfy  $\Delta_{n,m}(\bar{z}) \geq 0$ , while if  $\bar{z} \geq \Phi_{n,m}$  then any local maximum must satisfy  $\Delta_{n,m}(\bar{z}) \leq 0$ . Therefore either one of the following three cases must be true: (i)  $\Delta_{n,m}(\cdot) \geq 0$  on  $(-\infty, 0]$ , or (ii)  $\Delta_{n,m}(\cdot) \leq 0$  on  $(-\infty, 0]$ , or (iii)  $\Delta_{n,m}(\cdot)$  crosses 0 exactly once from above. Therefore there exists a  $T_{n,m}$  such that  $\Delta_{n,m}(q_0) \geq 0$  if and only if  $q_0 \leq -T_{n,m}$ , or equivalently the manager is better off employing  $m > n$  rather than  $n$  agents if and only if  $|q_0| \geq T_{n,m}$ . By noting that  $T_{n,m} = 0$  under case (i), and  $T_{n,m} = \infty$  under case (ii), the proof is complete. □

*Proof of Corollary 1.* Other things equal, the manager chooses the team size  $n \in \mathbb{N}$  to maximize her expected discounted profit at  $q_0$ ; i.e., she chooses  $n(|q_0|) = \arg \max_{n \in \mathbb{N}} \{F_n(q_0)\}$ . By noting that the necessary conditions for the Monotonicity Theorem (i.e., Theorem 4) of

Milgrom and Shannon (1994) to hold are satisfied, it follows that the optimal team size  $n(|q_0|)$  is (weakly) increasing in the project size  $|q_0|$ . □

*Proof of Proposition 6.* This proof is organized in 2 parts.

**Part I: Agents' Problem**

**(a) Formulation of the Agents' Problem**

To begin, fix the manager's budget  $B < U$  and the retirement state  $R$ . Then denote by  $\bar{J}(\cdot)$  each agent's expected discounted payoff when both agents carry out the project to completion together. Let us assume by convention that as soon as the project hits  $R$  for the first time, agent 2 will retire, and agent 1 will carry out the remainder of the project on his own. Upon completion of the project, agent  $i$  receives  $V_i$ , where  $V_1 + V_2 = B$ . The  $V_i$ 's will be chosen such that  $J_1(R) = J_2(R)$ ; i.e., the agents have the same expected discounted payoff when the project hits  $R$  for the first time. This will ensure that the agents' strategies before agent 2 retires are identical (which makes the analysis tractable). Therefore, denote by  $J_R(\cdot)$  the expected discounted payoff of each agent before agent 2 has retired. Note that  $\bar{J}(\cdot)$  and  $J_i(\cdot)$  are defined on  $(-\infty, 0]$ , while  $J_R(\cdot)$  is defined on  $(-\infty, R]$ .

Using (4),  $\bar{J}(\cdot)$  satisfies

$$r\bar{J}(q) = -c(f(\bar{J}'(q))) + 2f(\bar{J}'(q))\bar{J}'(q) + \frac{\sigma^2}{2}\bar{J}''(q) \text{ s.t. } \lim_{q \rightarrow -\infty} \bar{J}(q) = 0 \text{ and } \bar{J}(0) = \frac{B}{2}.$$

Because the state of the project  $q$  can drift back below  $R$  after agent 2 has retired,  $J_1(\cdot)$  and  $J_2(\cdot)$  satisfy

$$\begin{aligned} rJ_1(q) &= -c(f(J_1'(q))) + f(J_1'(q))J_1'(q) + \frac{\sigma^2}{2}J_1''(q) \text{ s.t. } \lim_{q \rightarrow -\infty} J_1(q) = 0 \text{ and } J_1(0) = V_1, \text{ and} \\ rJ_2(q) &= f(J_1'(q))J_2'(q) + \frac{\sigma^2}{2}J_2''(q) \text{ s.t. } \lim_{q \rightarrow -\infty} J_2(q) = 0 \text{ and } J_2(0) = B - V_1 \end{aligned}$$

on  $(-\infty, 0]$ , respectively. Observe that after agent 2 retires, his expected discounted payoff depends on the effort of agent 1 and on his net payoff  $V_2$  upon completion of the project. By using the same approach as used to prove Proposition 1 (i), it follows that  $J_1(\cdot)$   $\{J_2(\cdot)\}$  increases  $\{\text{decreases}\}$  in  $V_1$ , and  $J_1(\cdot)$  and  $J_2(\cdot)$  depend continuously on  $V_1$ . Moreover,  $J_1(R) > J_2(R) = 0$  if  $V_1 = B$ , and it is straightforward to show that  $J_1(R) < J_2(R)$  if  $V_1 = \frac{B}{2}$ . Therefore, by the intermediate value theorem, there exists a  $V_1 > \frac{B}{2}$  such that  $J_1(R) = J_2(R)$ .

Next let us consider  $J_R(\cdot)$ . Using (4),  $J_R(\cdot)$  satisfies

$$rJ_R(q) = -c(f(J'_R(q))) + 2f(J'_R(q))J'_R(q) + \frac{\sigma^2}{2}J''_R(q) \text{ s.t. } \lim_{q \rightarrow -\infty} J_R(q) = 0 \text{ and } J_R(R) = J_1(R),$$

where the second condition ensures value matching at  $q = R$ . Because  $J_1(\cdot)$  and  $J_2(\cdot)$  are *pinned down* independently of  $J_R(\cdot)$ , the above boundary conditions completely characterize  $J_R(\cdot)$ .

**(b)** Show that  $J_R(R) \leq \bar{J}(R)$ , and hence  $J'_R(q) \leq \bar{J}'(q)$  for all  $q \leq R$ .

Let  $D(q) = J_1(q) + J_2(q) - 2\bar{J}(q)$ , note that  $\lim_{q \rightarrow -\infty} D(q) = D(0) = 0$ , and  $D(\cdot)$  is smooth. Therefore either  $D(\cdot) \equiv 0$  on  $(-\infty, 0]$ , or  $D(\cdot)$  has at least one interior extreme point. Suppose the latter is true, and let us denote this extreme point by  $\hat{z}$ . Then  $D'(\hat{z}) = 0$  so that

$$\begin{aligned} rD(\hat{z}) &= -c(f(J'_1(\hat{z}))) + 2c(f(\bar{J}'(\hat{z}))) + 2[f(J'_1(\hat{z})) - 2f(\bar{J}'(\hat{z}))]\bar{J}'(\hat{z}) + \frac{\sigma^2}{2}D''(\hat{z}) \\ \Rightarrow rD(\hat{z}) &= -\frac{1}{2}\left\{2[\bar{J}'(\hat{z})]^2 + [J'_1(\hat{z}) - 2\bar{J}'(\hat{z})]^2\right\} + \frac{\sigma^2}{2}D''(\hat{z}). \end{aligned}$$

Suppose that  $\hat{z}$  is a maximum. Then  $D''(\hat{z}) \leq 0$ , and because the first term in the RHS is strictly negative, it follows that  $D(\hat{z}) < 0$ . This implies that any local maximum  $\hat{z}$  must satisfy  $D(\hat{z}) \leq 0$ , which leads me to conclude that  $D(q) \leq 0$  for all  $q$ . Moreover, because the inequality is strict, note that it cannot be case that  $D(\cdot) \equiv 0$  on  $(-\infty, 0]$ . Because  $J_R(R) = J_1(R) = J_2(R)$ , the result implies that  $J_R(R) \leq \bar{J}(R)$ . Finally, by applying Proposition 1 (i), it follows that  $J'_R(q) \leq \bar{J}'(q)$  for all  $q \leq R$ .

## Part II: Manager's Problem

### (a) Formulation of the Manager's Problem

To begin, denote by  $\bar{F}(\cdot)$  the manager's expected discounted profit when both agents carry out the project to completion together. Denote by  $F_1(\cdot)$  the manager's expected discounted profit when one agent carries out the project alone (i.e., after agent 2 has retired). Denote by  $F_R(\cdot)$  the manager's expected discounted profit taking into account that agent 2 will retire at the first time that the state of the project hits  $R$ . Note that  $\bar{F}(\cdot)$  and  $F_1(\cdot)$  are defined

on  $(-\infty, 0]$ , while  $F_R(\cdot)$  is defined on  $(-\infty, R]$ . Using (5),  $\bar{F}(\cdot)$  and  $F_1(\cdot)$  satisfy

$$\begin{aligned} r\bar{F}(q) &= 2f(\bar{J}'(q))\bar{F}'(q) + \frac{\sigma^2}{2}\bar{F}''(q) \text{ s.t. } \lim_{q \rightarrow -\infty} \bar{F}(q) = 0 \text{ and } \bar{F}(0) = U - B, \text{ and} \\ rF_1(q) &= f(J_1'(q))F_1'(q) + \frac{\sigma^2}{2}F_1''(q) \text{ s.t. } \lim_{q \rightarrow -\infty} F_1(q) = 0 \text{ and } F_1(0) = U - B, \end{aligned}$$

respectively. Finally, the manager's expected discounted profit before one agent is retired satisfies

$$rF_R(q) = 2f(J_R'(q))F_R'(q) + \frac{\sigma^2}{2}F_R''(q) \text{ s.t. } \lim_{q \rightarrow -\infty} F_R(q) = 0 \text{ and } F_R(R) = F_1(R),$$

where the second condition ensures value matching at  $q = R$ . Because  $F_1(\cdot)$  is determined independently of  $F_R(\cdot)$ , these boundary conditions completely characterize  $F_R(\cdot)$ .

**(b)** Show that there exists a  $\Theta_R \leq R$  such that  $F_R(q_0) \geq \bar{F}(q_0)$  if and only if  $\Theta_R \leq q_0 < R$ . First, let  $\Delta_1(q) = F_1(q) - \bar{F}(q)$ , and note that  $\lim_{q \rightarrow -\infty} \Delta_1(q) = \Delta_1(0) = 0$ , and that  $\Delta_1(\cdot)$  is smooth. As a result, either  $\Delta_1(\cdot) \equiv 0$  on  $(-\infty, 0]$ , or it has at least one interior extreme point. Suppose that the latter is true, and let us denote such extreme point by  $z^*$ . Then  $\Delta_1'(z^*) = 0$ , which implies that

$$r\Delta_1(z^*) = [f(J_1'(z^*)) - 2f(\bar{J}'(z^*))]\bar{F}'(z^*) + \frac{\sigma^2}{2}\Delta_1''(z^*).$$

It is straightforward to prove a result analogous to Theorem 2 (B): that there exists a threshold  $\Phi$  such that  $f(J_1'(z^*)) \leq 2f(\bar{J}'(z^*))$  if and only if  $z^* \leq \Phi$ . As a result  $\Delta_1(z^*) \leq 0$  if  $z^* \leq \Phi$ , while  $\Delta_1(z^*) \geq 0$  if  $z^* \geq \Phi$ . It follows that  $\Delta_1(\cdot)$  crosses 0 at most once from below.

Next, define  $\Delta_R(q) = F_R(q) - \bar{F}(q)$  on  $(-\infty, R]$ . Note that  $\lim_{q \rightarrow -\infty} \Delta_R(q) = 0$ ,  $\Delta_R(R) = \Delta_1(R)$ , and  $\Delta_R(\cdot)$  is smooth, where the second equality follows from the value matching condition  $F_R(R) = F_1(R)$ . Because  $\Delta_1(\cdot)$  crosses 0 at most once from below, depending on the choice of the retirement point  $R$ , it may be the case that  $\Delta_1(R) \stackrel{\leq}{\geq} 0$ .

Suppose  $\Delta_1(R) \geq 0$ . Then either  $\Delta_R(\cdot)$  increases in  $(-\infty, R]$ , or it has at least one interior extreme point. Suppose the latter is true, and let us denote such extreme point by  $\bar{z}$ . Then  $\Delta_R'(\bar{z}) = 0$  implies that

$$r\Delta_R(\bar{z}) = 2[f(J_R'(\bar{z})) - f(\bar{J}'(\bar{z}))]\bar{F}'(\bar{z}) + \frac{\sigma^2}{2}\Delta_R''(\bar{z}).$$

Recall from part I (c) of this proof that  $J'_R(q) \leq \bar{J}'(q)$  for all  $q \leq R$ , which implies that  $f(J'_R(\bar{z})) \leq f(\bar{J}'(\bar{z}))$ . It follows that  $\bar{z}$  must satisfy  $\Delta_R(\bar{z}) \leq 0$ . Because  $\Delta_1(R) \geq 0$ , it follows that there exists a threshold  $\Theta_R < R$  such that  $\Delta_1(q_0) \geq 0$  if and only if  $\Theta_R \leq q_0 < R$ . If  $\Delta_1(R) < 0$ , the same analysis yields that  $\Delta_R(\cdot)$  decreases in  $(-\infty, R]$ , and hence  $\Delta_1(q_0) \leq 0$  for all  $q_0 \leq R$ .

**(c) Conclusion of the Proof**

I have shown that as long as  $R$  is chosen such that  $F_1(R) \geq \bar{F}(R)$  (so that  $\Delta_1(R) \geq 0$ ), there exists a threshold  $\Theta_R < R$  such that  $F_R(q_0) \geq \bar{F}(q_0)$  for all  $q_0 \in [\Theta_R, R]$ . The last relationship implies that as long as the size of the project  $|R| < |q_0| \leq |\Theta_R|$ , the manager is better off implementing the proposed retirement scheme relative to allowing both agents to carry out the project to completion together. Finally, the requirement that  $R$  is chosen such that  $F_1(R) \geq \bar{F}(R)$  is equivalent to the requirement that if the project size were  $|q_0| = |R|$ , and the manager did not use a dynamic team size management scheme, she would be better off employing one instead of two agents. □

*Proof of Proposition 7.* In preparation, I first establish two Lemmas.

**Lemma 2.** *Consider a project undertaken by two identical agents who differ only in their final rewards such that  $V_1 > V_2$ . Also, suppose that effort costs are quadratic. Then  $\frac{d}{dq}[a_1(q) - a_2(q)] \geq 0$  for all  $q$ .*

*Proof of Lemma 2.* Observe that when effort costs are quadratic, then  $a_i(q) = J'_i(q)$ , so it suffices to show that  $D'_J(\cdot) = J'_1(\cdot) - J'_2(\cdot)$  is (weakly) increasing on  $(-\infty, 0]$ . First note that  $\lim_{q \rightarrow -\infty} D'_J(q) = 0$ , and from Proposition 1 (i), it follows that  $D'_J(q) > 0$  for all  $q$ . Fix  $z \leq 0$ , and let  $\bar{z} = \arg \max\{D'_J(q) : q \leq z\}$ . Clearly,  $\bar{z} > -\infty$ . Suppose that  $\bar{z}$  is interior. Then  $D''_J(\bar{z}) = 0$  and  $D'''_J(\bar{z}) \leq 0$ , and by using (8) we have that  $rD'_J(\bar{z}) = \frac{\sigma^2}{2}D'''_J(\bar{z}) \leq 0$ . However, this contradicts the fact that  $D'_J(\bar{z}) > 0$ , which implies that  $\bar{z} = z$ . Since  $z$  was chosen arbitrarily, this implies that  $D'_J(\cdot)$  is (weakly) increasing on  $(-\infty, 0]$ . □

**Lemma 3.** *Consider a project undertaken by two identical agents, and suppose that effort costs are quadratic. Consider the following two scenarios for the agents' compensation: (i)  $V_1 = V_2 = \frac{B}{2}$ , and (ii)  $V_1 = \frac{B}{2} + \epsilon > \frac{B}{2} - \epsilon = V_2$ . Then for all  $\epsilon \in (0, \frac{B}{2}]$  there exists a  $\Theta_\epsilon < 0$  such that the aggregate effort of the team is larger under asymmetric rewards (i.e., under scenario (ii)) if and only if  $q \geq \Theta_\epsilon$ .*

*Proof of Lemma 3.* First let us denote the expected discounted payoff function of the agents under asymmetric compensation by  $J_1(q)$  and  $J_2(q)$ , respectively, and let us denote the expected discounted payoff function of the agents under symmetric compensation by  $J_S(q)$ . Because effort costs are quadratic,  $a_i(q) = J'_i(q)$ . Observe that we are interested in comparing  $2a_S(q)$  and  $a_1(q) + a_2(q)$ , or equivalently  $2J'_S(q)$  and  $J'_1(q) + J'_2(q)$  on  $(-\infty, 0]$ . Let us define  $M(q) = 2J_S(q) - J_1(q) - J_2(q)$ . By noting that  $\lim_{q \rightarrow -\infty} M(q) = M(0) = 0$  and  $M(\cdot)$  is smooth on  $(-\infty, 0]$ , it follows that either  $M(\cdot) \equiv 0$ , or it has at least one interior global extreme point. Suppose the latter is true and let us denote that extreme point by  $z^*$ . By using (4), and the facts that  $f(x) = x$  and  $c(f(x)) = \frac{x^2}{2}$ , it follows that

$$rM(z^*) = \frac{1}{2} \left[ 6(J'_S(z^*))^2 - 2(J'_1(z^*) + J'_2(z^*))^2 + (J'_1(z^*))^2 - (J'_2(z^*))^2 \right] + \frac{\sigma^2}{2} M''(z^*) .$$

Because  $z^*$  is an extreme point,  $M'(z^*) = 0$  implies that  $J'_S(z^*) = \frac{J_1(z^*) + J_2(z^*)}{2}$ . By substituting into the above equality and simplifying the terms, we have

$$rM(z^*) = \frac{1}{4} [J'_1(z^*) - J'_2(z^*)]^2 + \frac{\sigma^2}{2} M''(z^*) .$$

Suppose that  $z^*$  is a global interior minimum. Then the facts that  $M''(z^*) \geq 0$  and  $J'_1(z^*) > J'_2(z^*)$  (which follows from Proposition 1 (i)), imply that  $M(z^*) > 0$ . However, this contradicts the fact that  $M(0) = 0$ , which implies that  $z^*$  must be a maximum and  $M(q) \geq 0$  for all  $q$ . Moreover, because  $J_1(z^*) > J_2(z^*)$ , note that it cannot be the case that  $M(\cdot) \equiv 0$ .

Now suppose that  $M(\cdot)$  has more than one extreme points. Then there must exist a local maximum  $z^*$  followed by a local minimum  $\bar{z} > z^*$ . This implies that  $M''(z^*) \leq 0 \leq M''(\bar{z})$ , and by Lemma 2,  $0 \leq J'_1(z^*) - J'_2(z^*) \leq J'_1(\bar{z}) - J'_2(\bar{z})$ . These equalities imply that  $M(z^*) \leq M(\bar{z})$ , which contradicts the assumption that  $z^*$  is a maximum and  $\bar{z}$  is a minimum. Hence  $M(\cdot)$  has a global maximum on  $(-\infty, 0]$  and no other local extreme points. Therefore there exists a  $\Theta_\epsilon < 0$  such that  $M'(q) \geq 0$  if and only if  $q \leq \Theta_\epsilon$ . □

To begin, let us denote the manager's expected discounted profit by  $F_0(q)$  and  $F_\epsilon(q)$  under the symmetric (i.e.,  $(\frac{B}{2}, \frac{B}{2})$ ) and the asymmetric (i.e.,  $(\frac{B}{2} + \epsilon, \frac{B}{2} - \epsilon)$ ) compensation scheme, respectively. Moreover, let us denote the expected discounted payoff of each agent by  $J_S(\cdot)$ ,  $J_1(\cdot)$ , and  $J_2(\cdot)$ , where the subscripts follow the convention from Lemma 3. Next, let  $\Delta_\epsilon(q) = F_0(q) - F_\epsilon(q)$ , and observe that  $\lim_{q \rightarrow -\infty} \Delta_\epsilon(q) = \Delta_\epsilon(0) = 0$ . Therefore, either  $\Delta_\epsilon(\cdot) \equiv 0$ , or  $\Delta_\epsilon(\cdot)$  has at least one interior global extreme point. Suppose the latter is true,

and let us denote that extreme point by  $\bar{z}$ . By using (5) and the fact that  $\Delta'_\epsilon(\bar{z}) = 0$ , it follows that

$$r\Delta_\epsilon(\bar{z}) = [2J'_S(\bar{z}) - J'_1(\bar{z}) - J'_2(\bar{z})]F'_0(\bar{z}) + \frac{\sigma^2}{2}\Delta''_\epsilon(\bar{z}).$$

From Lemma 2, we know that there exists a threshold  $\Theta_\epsilon$  such that  $2J'_S(q) \geq J'_1(q) + J'_2(q)$  if and only if  $q \leq \Theta_\epsilon$ , and from Theorem 3 (ii) that  $F'_0(q) > 0$  for all  $q$ . It follows that  $\bar{z}$  is a global maximum if  $\bar{z} \leq \Theta_\epsilon$ , while it is a global minimum if  $\bar{z} \geq \Theta_\epsilon$ . Moreover, any local extreme point  $\bar{z} \leq \Theta_\epsilon$  must satisfy  $\Delta_\epsilon(\bar{z}) \geq 0$ , while any local extreme point  $\bar{z} \geq \Theta_\epsilon$  must satisfy  $\Delta_\epsilon(\bar{z}) \leq 0$ . Moreover, because  $2J'_S(q) > J'_1(q) + J'_2(q)$  for at least some  $q$ , and  $F'_0(q) > 0$  for all  $q$ , it cannot be the case that  $\Delta_\epsilon(\cdot) \equiv 0$ . Therefore, either one of the following three cases must be true: (i)  $\Delta_\epsilon(\cdot) \geq 0$  on  $(-\infty, 0]$ , (ii)  $\Delta_\epsilon(\cdot) \leq 0$  on  $(-\infty, 0]$ , or (iii)  $\Delta_\epsilon(\cdot)$  crosses 0 exactly once from above. Hence, there exists a  $T_\epsilon$  such that  $F_0(q_0) \geq F_\epsilon(q_0)$  if and only if  $q_0 \leq -T_\epsilon$ , or equivalently if and only if  $|q_0| \geq T_\epsilon$ . □

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